PLANE CONSTRUCTION FIELD THEORY

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INTRODUCTION

This thesis contains the results of an investigation into the construction operations used for the ordinary process of construction in some elementary construction geometries, as well as some applications to a theory concerning the complexes of objects constructible from a set of given objects. Such a complex will be called a *construction field*; the set of given objects will be called its *basis*.

When we want to consider geometry, analogously to algebraical systems, as an abstract mathematical system composed of some classes of *objects* of different kinds, and of a set of *relations*, while the constructive actions play the part of *operations* which generally connect objects of different kind, it is immediately clear that a geometry will have a much more complicated structure than an algebra has. The number of operations will commonly be larger, the symbolical notations representing them will be more intricate, so that the formulas describing a certain construction process generally will be circumstantial. Nevertheless we shall make an attempt at collecting the operations in geometry systematically and representing them by symbols, distinguishing between postulates and theorems.

Approaching the problem in ordinary Euclidean geometry we have considered a series of elementary construction geometries, beginning with the linear construction geometry in projective plane, in which, gradually, more concepts are defined and more operations are postulated. The theory will mainly be limited to the classical instruments of construction, ruler and compasses. Only some sections are devoted to the instrument "right square" (German: "rechter Zeichenwinkel").

A special difficulty occurs in cases where a certain operation (e.g., the intersection of two circles) produces two objects simultaneously, that are equivalent with respect to the preceding construction, but perhaps non-equivalent with respect to the solution of a certain construction problem (bivalent operations). Already in 1909 H. TIETZE [1] discerned these difficulties. He separated a certain domain of construction from the total set of classically constructible objects in Euclidean plane, which does not contain this ambiguity. In some later papers [2], [3], he returned to the subject. In the latter TIETZE gave a slight indication concerning the direction in which he supposed the final solution of the problem was to be found. Indeed, it is a curious thing that, as far as it is known to us, nobody has taken up the subject again. The theorems of TIETZE find their places in the construction field theory developed here.

The operations in a construction geometry will be divided into (*a*) *active operations*, producing new objects from some given ones, (*b*) *decision operations*, giving a decisive answer with regard to certain relations existing between some objects, (*c*) *selection operations*, fixing certain representatives of a set of equivalent given or constructed objects, and (*d*) *adjunction operations*, adding to a figure certain new objects that cannot be constructed from that figure by means of the available active operations.

It is the very introduction of decision operations which makes it possible to solve the problem of non-ambiguity in Euclidean construction geometry mentioned above.

The possibility of some of the decision operations will be *postulated*, which implies, that the construction geometries dealt with are assumed to constitute decidable systems with regard to the relations corresponding with the decision operations in question. The introduction of the decision operations as postulates is due to logical considerations. To explain this we give an example.

Let be given a geometry G in which the following axioms hold:

I. Let a, b be two points of G, then either $a = b$ or $a \neq b$,

II. There exists exactly one line $L_1(a, b) = \bar{p}$ passing through a and b, if and only if $a \neq b$.

By virtue of these axioms we may introduce the active operation postulate: The active operation L_1 producing the line $L_1(a, b)$ is applicable to the points a and b, if and only if $a \neq b$ is provable.

The performability of this active operation in a certain special case is logically equivalent with the possibility of the decision whether the points in question are distinct. Wishing to require that it will be possible to construct the line passing through *any* couple of distinct points it will be convenient to introduce the decision operation *Dpp* as a postulate: It is possible to decide whether two points are distinct or not. Perhaps it seems a needless duplication when such a decision operation is placed beside the active operation. But the method satisfies quite well and gratifies the logical intuition; besides, we shall meet decision operations that are not directly connected with active operations, so that, in those cases, their reason of existence is ensured undoubtedly.

We should like to make another remark motivating the use of the decision operations and explaining the relations between the decision operations and the adjunction operations in connection with the possible provability of certain relations existing between the objects of a geometry. Especially with regard to adjoined objects the decision operations (as postulates) play an important part. We fear that, on this subject, we cannot be clear enough in this introduction, so we refer to Chapter I, 3, pp. 11 ff.

Considerable use has been made of the concept of *construction field*, this being the set of objects in a certain geometry constructible from a certain set of fixed given objects (*basis*), by means of the active operations available, and of the concept of *automorphism of a construction field*. In particular we are interested in automorphisms leaving the basis objectwise fixed. In non-ordered ruler and compasses construction geometries some theorems will be derived concerning the connection between the automorphism group of a construction field and the Galois group of the coordinate field belonging to that construction field, over a certain subfield (Chapter V).

The subject of this thesis may have some significance for axiomatics of geometry. When founding a geometry axiomatically certain kinds of objects and relations will have to be introduced as undefined. The selection of these fundamental elements is a question of arbitrariness; efficiency, elegance, and mathematical-philosophical considerations being the leading ideas. Moreover in classical geometry the abstracts of drawing operations that are actually practicable stand in close connection with fundamental axioms. The philosophical investigation in the structure of this connection is, indeed, an interesting problem; the insight with respect to the connection between axioms and construction postulates may be extended, the possibility of a modern, abstract view on elementary geometry may be favoured by a systematical classification and a formal notation of the construction operations.

The objects of a geometry are considered *abstractly*. The geometrical building is assumed to be raised axiomatically. Most of the geometries dealt with are of a well known type so that the axioms in question shall not be mentioned; there are only two exceptions dealt with in Chapters IV en V Only those geometries are considered that can be brought into a one-to-one correspondence with certain analytical geometries over fields with certain characteristic. The characteristic of the field will also be called the characteristic of the geometry.

A separate chapter has been devoted to the MOHR-MASCHERONI constructions (constructions with compasses only) in Euclidean plane. We shall prove, among other things, that ARCHIMEDES' axiom is necessary for the validity of the chief MOHR-MASCHERONI theorem when we restrict ourselves to the use of the ordinary means of construction used in the construction geometry of compasses. In order that the theorem will also hold in non-Archimedean geometries certain *adjunction operations* may be admitted (Chapter VIII, 4).

In the first chapter one will find an explanation of the main concepts used throughout the book. At the end of the book some lists of symbols used are added.

Finally we remark that it was elementary geometry which, for many centuries, gave mankind the idea that mathematics had its foundation in an absolute Truth existing outside human thought. The development of science in the last few ages has taught us that mathematics and truth have little to do with each other. It is especially in elementary geometry that we again observe the distance between mathematics and truth when introducing a set of decision postulates properly as axioms and not on the ground of provable theorems. We do not ask *how* the decisions relating to the matter in question must be made. Indeed, truth is not important at all; what one *thinks* to be true is the only important thing. Therefore, mathematics is true, because it lives in the human mind only and explicitly declares not to be "truth"

Survey of construction geometries dealt with.

1. *Linear construction geometry in projective plane.* Linear objects: points and lines. Relations: coincidence and non-coincidence of

points, lines; incidence and excidence of point and line. Ruler constructions. (Chapter II).

2. *Linear construction geometry in affine plane.* Introduction of the relation: parallelism, and its negation. Introduction of the corresponding operations. (Chapter III).

3. *Construction geometry of reflections.* Introduction of the objects: circles, and of the relation: congruence. Univalent operations with compasses. (Chapter IV).

4. *Construction geometry of normals* (*i*). Operations with the "right square" producing the line passing through a given point normal to a given line (First using of the instrument). (Chapter IV).

5. *Construction geometry in a congruence geometry using restricted branching operations.* Introduction of a certain bivalent operation with compasses. (Chapter V).

6. *Construction geometry built in* HILBERT *geometry using restricted branching operations.* Introduction of the relation: betweenness, and of the corresponding decision operations. (Chapter VI).

7. *Construction geometry of ruler and compasses in Euclidean plane.* Full use of branching operations and of decision operations of order. (Chapter VII).

8. *Construction geometry of normals* (*ii*). Continuation of 4. Second using of the instrument "right square". No use of compasses. (Chapter VII).

9. MOHR-MASCHERONI *construction geometry in Euclidean plane.* Use of compasses only. Necessary introduction of ARCHIMEDES' axiom. Introduction of a certain enumerable adjunction process in non-Archimedean geometry. (Chapter VIII).

Moreover some smaller subjects have been included: in some special cases the influence of a limited use of certain operations to the structure of construction fields and operation sets has been investigated.

A generalisation of the theory and an extension to n-dimensional construction geometries seems possible.

CHAPTER I

GENERAL CONCEPTS

1. Construction operations.

Definition. *A geometry G is a mathematical system composed of some classes* K_1, K_2, of *abstract elementary objects* (points, lines, circles, .) and a set R of *abstract elementary relations* (coincidence, incidence, parallelism, normality, betweenness, .) regarding these objects; the existence of the elementary objects and relations, and the properties of G are based on a set of fundamental statements (*axioms*).

New kinds of objects, such as point couples, line couples, triangles, parallelograms, squares, etc., and of new relations, such as "being the centre of the circumscribed circle of a triangle", etc., may be defined. (*Derived* objects, *derived* relations.)

We shall only consider geometries that either can be or have been brought into a one-to-one correspondence with analytical geometries over fields with certain properties concerning characteristic, possibility of operations $\vartheta \to \sqrt{1 + \vartheta^2}$, or $\vartheta \to \sqrt{\vartheta}$, reality, etc., in the usual manner.

These fields are assumed sufficiently large so that any proper subfield may be extended unlimitedly by suitable algebraic or transcendental adjunctions.

Definitions. Let Δ be a field and let α, β be two elements of Δ. If it is possible to decide whether we have $\alpha = \beta$ or $\alpha \neq \beta$, we shall say that the establishment of the statement, whether $\alpha = \beta$ or $\alpha \neq \beta$ is the case, is *the application of the decision operation* D^0 *to the elements* α *and* β.

An algebraic statement concerning a finite number of elements α_1, ., α_n of Δ that can be reduced to a finite set of polynomial expressions $\varphi_\sigma(\alpha_1, \quad \alpha_n) = 0$, $\sigma = 1$, m, with coefficients in the prime field Π contained in Δ, is called a 0-*relation* existing between the elements α_1, α_n.

The set of decision operations D^o required to establish whether certain elements satisfy a certain 0-relation or not, is called a 0-*decision.*

0-relations and -decisions will also be called *relations and decisions of the first category.*

Let Δ be an ordered or a partially ordered field, and let α, β be two distinct elements of Δ. If it is possible to decide whether we have $\alpha > \beta$ or $\alpha < \beta$, we shall say that the establishment of the statement, whether $\alpha > \beta$ or $\alpha < \beta$ is the case, is *the application of the decision operation* $D\gtrless$ *to the elements* α *and* β.

An algebraic statement concerning a finite number of elements $\alpha_1, \quad ,\alpha_n$ of Δ that can be reduced to a finite set of polynomial expressions $\varphi_\sigma (\alpha_1, \quad , \alpha_n) > 0$, $\sigma = 1, \quad , m$, with coefficients in Π, is called a $\gtrless$-*relation,* existing between the elements $\alpha_1, \ldots, \alpha_n$.

The set of decision operations $D\gtrless$ required to establish whether certain elements satisfy a certain $\gtrless$-relation or not, is called a $\gtrless$- *decision.* $\gtrless$-relations and $\gtrless$-decisions will also be called *relations and decisions of the second category.*

Definition. A set of objects of G, finite or infinite, is called a *figure.*

The elementary objects considered are *points,* $a, b, c, \quad$, *lines,* $\bar{a}, \bar{b}, \bar{c}, \quad$, and *circles,* $\bar{\bar{a}}, \bar{\bar{b}}, \bar{\bar{c}},$

Lines and circles will not be considered as point sets. If a figure F contains a line $\bar{a}$, while $\bar{a}$ is incident with a certain point a, then F need not necessarily contain the point a too.

The relations that, in the analytical image of G, correspond with one or more algebraic 0-relations, will also be called 0-*relations.* We mention:

coincidence of two points a and b, written $a = b$ or $b = a$;
coincidence of two lines $\bar{a}$ and $\bar{b}$, written $\bar{a} = \bar{b}$ or $\bar{b} = \bar{a}$;
coincidence of two circles $\bar{\bar{a}}$ and $\bar{\bar{b}}$, written $\bar{\bar{a}} = \bar{\bar{b}}$ or $\bar{\bar{b}} = \bar{\bar{a}}$;
incidence of point a *and line* $\bar{a}$, written a inc $\bar{a}$ or $\bar{a}$ inc a;
incidence of point a *and circle* $\bar{\bar{a}}$, written a inc $\bar{\bar{a}}$ or $\bar{\bar{a}}$ inc a;
parallelism of two distinct lines $\bar{a}$ *and* $\bar{b}$, written $\bar{a} \parallel \bar{b}$ or $\bar{b} \parallel \bar{a}$;
normality of two distinct lines $\bar{a}$ and $\bar{b}$, written $\bar{a} \perp \bar{b}$ or $\bar{b} \perp \bar{a}$.

The respective negations of these relations are not considered as relations themselves: $a \neq b$ or $b \neq a$, $\bar{a} \neq \bar{b}$ or $\bar{b} \neq \bar{a}$, $\bar{\bar{a}} \neq \bar{\bar{b}}$ or $\bar{\bar{b}} \neq \bar{\bar{a}}$, a exc $\bar{a}$ or $\bar{a}$ exc a, a exc $\bar{\bar{a}}$ or $\bar{\bar{a}}$ exc a; for the negations of $\parallel$ and $\perp$ no special symbol will be used.

Furthermore, *congruence* will be considered as a 0-relation that may exist between two (non-elementary) objects, such as point couples, line couples, distances, angles, etc.; the symbol for congruence is $\simeq$. Possessing *similar orientation* will be defined as a 0-relation for isosceles right-angled triangles (non-elementary objects); an oriented isosceles right-angled triangle will be denoted by $\overrightarrow{a, b, c}$ or $\overleftarrow{c, b, a}$, b being the vertex of the right angle.

The relations that, in the analytical image of G, correspond with one or more algebraic $\gtrless$-relations, will similarly be called $\gtrless$-*relations*.

Some of them will be mentioned:

betweenness of three distinct collinear points: "b between a and c" will be denoted by $/\,a,b,c\,/$ or $/\,c,b,a\,/$; *position of two points on similar or opposite sides of a line*; *position* of a point *inside or outside* a circle; *intersection* and *non-intersection* of line and circle, or of two circles.

Possessing *similar* or *opposite orientation* is a $\gtrless$-relation that can be defined for arbitrary proper triangles; an oriented triangle will be denoted by $\overrightarrow{a,b,c}$, or $\overrightarrow{b,c,a}$, etc.

Remark. In the present chapter it will be convenient to use ordinary letters a, b, c, in order to denote elementary objects, either points, lines, or circles, in those cases where misunderstanding is excluded.

Let it be possible to introduce a set V of *constructive operations* performable in G; these operations may connect objects of different classes.

Definition. The complex of properties of G with respect to the operations of V is called the *construction geometry* (G, V).

V will consist of three kinds of operations: decision operations, active operations, and selection operations. A fourth kind, the adjunction operations, will be introduced later on; they will not belong to V

(1) Decision operations.

Analogously with our remarks on decisions in fields we give the following definition:

The establishment whether a certain relation exists between certain objects of a geometry G is called *the application of a decision operation concerning that relation, to the objects in question.*

Corresponding with each relation in G a decision operation may be introduced.

Examples. D^{pp}, decision operation establishing whether two points coincide or not; D^{pl}, decision operation establishing whether a point and a line are incident or excident; D^{par}, decision operation establishing whether two distinct lines are parallel or not; D^{pc}, decision operation establishing whether a point and a circle are incident or excident; $D^{p\cdot p\cdot p}$, decision operation establishing which of three distinct collinear points is between the others; $D^{p\cdot c}$, decision operation establishing whether a point being excident with a circle is inside or outside that circle, etc.

Again we will speak about 0-decisions and 0-decision operations (D^{pp}, D^{par}, .) and about $\gtrless$-decisions and $\gtrless$-decision operations ($D^{p\cdot p\cdot p}$, $D^{p\cdot c}$, .), also called decisions and decision operations of the *first* and of the *second category*, respectively.

(2) Active operations.

Definition. Let $a, b,$ $, k$ be a finite number of elementary objects of G, and let it have been either proved or postulated that there are *at most a finite number of elementary objects* g_i (of a certain class) in G, so that certain prescribed 0-relations R_ν $(g_i, a, b,$ $, k)$ exist between each g_i and the objects $a, b,$ $, k$, *the operation by which a figure* F (*containing the objects* $a, b,$ $, k$) *is extended when adding all* g_i, *is called an active operation, applied to the objects* $a, b,$ $, k$.

If an active operation produces exactly *one* new object, it is called *univalent*; if it produces *two* objects simultaneously, it is called *bivalent*.

We point out that *we shall only be concerned with univalent and bivalent active operations applied to two elementary objects*. It is a curious thing that such operations suffice to solve those construction problems that are solvable in the usual elementary construction geometries dealt with in this book.

The active operations will be denoted by symbols such as P_1, P_2, L_1, L_2, C_1, C_2, according as the resulting objects are *points, lines*, or *circles*, respectively. As usual in algebra too, both operation and resulting object will be denoted by the same symbol. When describing a certain construction process by means of formulas compiled of these symbols, the possibility of substitution into these formulas is obtained.

Example. $L_1(a,b)$ represents the operation by which the figure of two distinct points a and b is extended with the line passing through a and b, as well as the line through a and b itself.

In some construction geometries we speak about a certain object without having the disposal of the corresponding active operation. In such a case we shall denote the object in question by the usual operation symbol with an index ${}^{(o)}$ at the top. E.g., in MOHR-MASCHERONI construction geometry the line passing through two distinct points a and b will be denoted by $L_1^{(o)}(a,b)$.

(3) Selection operations.

Definition. Let F be a figure in G and let R be a set of relations. *Two objects a_1 and a_2 of G are called $F(R)$-equivalent if and only if all relations of R that exist between a_1 and the objects of F also exist between a_2 and the objects of F, and conversely.*

Two objects a_1 and a_2 of G are called F-equivalent if and only if all relations (possible in G) that exist between a_1 and the objects of F also exist between a_2 and the objects of F, and conversely.

If no ambiguity is possible we shall also use the mere word "equivalent"

Corollary. Two equivalent objects are of the same class. The equivalence relation is reflexive, symmetric, and transitive. It is a logical and not a geometrical relation.

The general selection postulate has a logical character and says that *it is possible to select an object arbitrarily out of a set of equivalent objects.*

The *selection operation* is denoted by S_1; the use of the symbol can be illustrated by an example: Let a & b & & k represent a set of equivalent points; then S_1 (a & b & . & k) $= u$ means: an arbitrary selection out of the point set composed of a, k gives us the point u.

Equivalent objects mostly will be joined by the symbol &. Nevertheless the symbol will also be used to join objects, when the question whether they are equivalent or not is left as it is.

We will introduce another selection symbol S_2 which nevertheless does not represent a new operation. The symbol can be used in order to combine the operations of decision and selection in some frequent special cases.

Let F_1 and F_2 be two figures in G and let a_1 & a_2 be two F_1-equivalent objects. Let it have been proved that at least one of the objects a_1 & a_2 satisfies a certain prescribed relation R_o with respect to certain objects of F_2. Then the notation (with an addition in "prose", for the sake of clearness) S_2 (a_1 & a_2) $=$ *a so that the relation* R_0 *exists between a and* b_1, b_2, b_k, will represent both decision and selection together. The notation is used when it has been proved that at least one of the objects a_1 and a_2 satisfies the relation required, or that but one of them satisfies it. In the latter case it is not exactly by selection that the object a is determined, but the entire process of decision and nomination is represented by the symbol S_2.

Before passing on to the discussion of the fourth kind of operations, the adjunction operations, we will introduce the concept of construction field.

2. Construction fields (First kind).

Let G be a geometry and let V be the set of construction operations of the three types mentioned above, the applicability of which is *postulated* in G; let B be a set of objects of G.

Definition. *The set of objects of G attainable by successive application of the operations of V to the objects of B and to the objects obtained, in a finite number of steps, will be called the construction field in the geometry G under the operations of V upon the basis B, or the* G_V*-field upon the basis B, or the construction field* G_V *(B).*

G_V (B) is the *constructively closed* set of objects in G upon B under the operations of V

After the introduction of the adjunction operations a second kind of construction field will be defined.

For the sake of convenience we shall restrict ourselves to bases composed of *points*. Generally, we consider the points as primary objects. Lines and circles will only occur when constructed from points by means of the active operations in question, except in some particular cases which will be mentioned later on.

3. Adjunction operations. Second kind of construction field.

General adjunction postulate. Let F be a figure in a geometry G, and let g be a point of G not belonging to F. *Then there exists an adjunction operation A, which when applied to F and g, adjoins g to F, forming a new figure* $F' = [F,g]$.

We will generally restrict ourselves to the adjunction of *points*.

The adjunction operations can be distinguished into various kinds. To explain this, we remark first that it is possible to consider construction geometry from two different points of view. From the *geometrical* (axiomatical) point of view we are interested in the possible solution of construction problems: the problems of constructing certain figures in G that satisfy certain conditions expressed in geometrical terms, by the use of certain construction operations, in a finite number of steps. Consequently only finite figures are considered. When performing a certain construction it occurs that a point is added to a figure that cannot be constructed by the use of the available active operations, e.g., a first point a is adjoined to the empty figure, a second point b distinct from a is adjoined to the figure a, the operation L_1 applied to a,b produces the line $L_1(a,b) = \bar{a}$, a third point c distinct from a and b and incident with $\bar{a}$ is adjoined, a fourth point d excident with $\bar{a}$ is adjoined to the figure formed by a, b, c, $\bar{a}$, etc.

This kind of adjunctions will be called *geometrical adjunctions*. From the geometrical point of view we are lead to the introduction of the *adjunction postulate*:

A^g: *Let be given a finite figure F; if it is known that G contains a point g that satisfies and/or does not satisfy one or more prescribed incidence relations and/or $\gtrless$-relations with certain objects of F, and that cannot be constructed from the objects of F by the use of the operations of a certain operation set V, there exists an adjunction operation A^g (in the construction geometry (G,V)) which extends F by the adjunction of such a point.*

Examples: adjunction of a point excident with any object of a given figure F, adjunction of a point incident with a certain line, excident with a certain line, inside or outside a certain circle, incident with (the circumference of) a certain circle, of F, etc.

There are two kinds of geometrical adjunction operations that will get a special name.

If the adjoined point g is a *"constant" point* in G (a fixed point in G), i.e. a point given uniquely in G, the adjunction of g to F will be called a *determinate adjunction*; the corresponding operation will be denoted by the symbol A^d. g will be called a *determinate point* over F. We shall need some determinate adjunction operations in reflection geometry (Chapter IV) and in MOHR-MASCHERONI construction geometry (Chapter VIII).

In the second place we want to consider certain adjunctions in a geometry G where *order* has been defined, possibly not with respect to all objects but in any case with respect to all objects of an ordered subgeometry G' of G; G itself may be called a *partially ordered geometry* in that case. Let B be a basis of the construction field $G_V(B)$ under the operations of the set V; let g be a point adjoined to B under the condition that *no* $\gtrless$*-decision operation concerning a* $\gtrless$*-relation between* g *and any object of* $G_V(B)$ *will be applicable.* In that case the adjunction of g to B will be called a *free adjunction*; g will be called a *free* point over B. The corresponding adjunction operation will be denoted by the symbol A^f.

If $A^f(B,g) = B'$, we may say that $G_V(B')$ is a partially ordered construction field. (Cf. Ch. VI, 3, p. 87).

When performing a certain construction process we want to take the geometrical point of view. This means that only geometrical adjunction operations are allowed. When using an analytical representation for geometry we shall denote the coordinates of the adjoined points by "parameters" (This notion will be defined later on). Presently we will distinguish between transcendental, semitranscendental, and algebraic adjunctions. These notions belong to the analytical language; in the geometrical construction process we will not use them. The question whether a geometrically adjoined point is transcendental, semitranscendental, or algebraic over a certain figure F cannot be decided in general, because the only available means are the decision operations. If an adjoined point is algebraic over a figure F, this can be established in certain cases by using a finite though unknown number of active and decision operations. If an adjoined point is transcendental or semitranscendental over F, this cannot be established within a finite number of steps, except in some special cases where F is very small, e.g., consisting of only one point. Consequently we may say that it is not possible to decide whether an adjoined point is transcendental, semitranscendental, or algebraic over a certain given figure F.

From the *analytical* point of view all elementary objects of G are represented by coordinates or by equations with coefficients, so by certain sets of elements of a certain field. Now also infinite figures may be considered; construction fields may be such figures. (The structure of construction fields will be described with analytical means). All objects constructible from a given basis B can be determined in analytical terms.

Let F be a figure in G and let Δ_S be the smallest field containing the coordinates of the points of F with respect to some coordinate system S. Let t be a point adjoined to F; its coordinates (with respect to the coordinate system in question) will be represented by parameters (τ_1, τ_2). Form the fields $\Delta_S(\tau_1)$, $\Delta_S(\tau_1, \tau_2)$ by successive adjunction of τ_1 and τ_2 to Δ_S.

Definitions. The point t is called a *transcendental point over F of degree $\leqslant i$*, if the transcendence degree of $\Delta_S(\tau_1, \tau_2)$ over Δ_S is $\leqslant i$ for every coordinate system S.

t is called a *transcendental point over F of degree i*, if it is a transcendental point over F of degree $\leqslant i$ and not of degree $\leqslant i$-1.

t is called a *transcendental point over F*, if it is a transcendental point over F of degree 2.

t is called a *semitranscendental point over F*, if it is a transcendental point over F of degree 1.

t is called an *algebraic point over F*, if it is a transcendental point over F of degree 0.

In an analytical treatment of the geometry we shall be able to speak about transcendental, semitranscendental, and algebraic adjunctions of a point t to a figure F, according as t is a transcendental, semitranscendental, or algebraic point over F. We shall not denote them by separate symbols; this kind of adjunctions will play a part in the analytical investigation of the construction fields.

Let G be a geometry, V a set of active, decision, and selection operations, the possibility of which is postulated in G. Let B be a fixed basis in G. We assume that the points of the construction field $G_V(B)$ can be characterized by the fact that their (inhomogeneous) coordinates with respect to some coordinate system belong to some complex of elements Φ, a ring, or a field, or a field with some restriction, or something else. The basis B is assumed to contain at least the points (0,0) and (1,0), if B contains at least two distinct points; if B only consists of one point, this point will be the point (0,0). If the geometry is given analytically, it is assumed that a coordinate transformation has taken place before, so that this condition is fulfilled. Transcendence of a point over B will be considered with regard to the smallest field Δ including Φ.

Let B be called B_0 and let t_k be a point transcendental over B_{k-1}, $k = 1, 2, \qquad$ and let t_k be adjoined to B_{k-1} forming the basis B_k.

Definition. *If it is possible to determine a number n in such a manner* *that the construction field* $G_V(B_{n-1})$ *is identical with the subset of* $G_V\ (B_n)$ *consisting of those objects that are independent of the adjoined point* t_n, *we define the adjunctive construction field* $G_{V(A)}\ (B_o)$ *as the set of objects of* $G_V(B_n)$ *that are independent of all adjoined points* t_k, $k = 1, \quad n$. (Second kind of construction field).

Example. Consider in affine geometry A the ruler operation set L consisting of the operations L_1 (line passing through two distinct points), L_2 (producing the line parallel to a given line $\bar{a}$ and passing through a given point a, a exc $\bar{a}$), P_1' (point of intersection of two distinct non-parallel lines), D^{pp} (decision whether two points coincide or not), D^{ll} (decision whether two lines coincide or not), S_1 (selection operation). Let a basis B consist of two points, z (0,0) and e_y (1;0); then $A_L\ (B)$ consists of these points and the line connecting them. But $A_{L(A)}\ (B)$ also contains the points $p(\pi,0)$, $\pi \in \Pi$, Π being the prime field contained in the field underlying our affine geometry. So $A_L\ (B) \neq A_{L(A)}\ (B)$ in the case $p \neq 2$ (p = characteristic of Π).

The adjunctive construction fields have been introduced by considering transcendental adjunctions. It is also possible to introduce suchlike notions when admitting only semitranscendental adjunctions (of a certain prescribed type), or when admitting both semitranscendental and transcendental adjunctions. Then special difficulties may occur concerning the concept "independence," because possible singularities must be excluded. Such questions will not be dealt with.

Transcendental, semitranscendental, and algebraic adjunctions are not considered as geometrical notions. When performing a certain construction process only geometrical adjunctions will be used. Sometimes it is possible to determine whether a geometrical adjunction is transcendental; the adjunction of one point to the empty basis may be defined as transcendental; the adjunction of a second point is transcendental according to the definition, etc., but generally it need not be possible, nor necessary, to determine the analytical character of a geometrical adjunction. For, it will often occur that a certain object of $G_{V(A)}\ (B)$ that does not belong to $G_V\ (B)$ is constructible in a construction field $G_V\ (B')$, where B' arises from B by the adjunction of a semitranscendental or algebraic point. Of course, the construction of the objects in question may not be described in such a manner as if the adjoined point were transcendental: then the construction may fail in the case that a certain algebraic adjunction has taken place. When describing such constructions as meant above, we must speak a geometrical language, using geometrical adjunction operations; the decision operations will then play an important role.

Nevertheless it will be convenient to speak about transcendental adjunctions in geometrical language too. We define: a geometrical adjunction operation will be called a geometrically *transcendental adjunction operation* if the adjoined point *may* be a transcendental point.

With $V(A)$, $L(A)$, etc., will be meant the operation set V, L, etc., extended with the set A of adjunction operations, but under the restriction that only those objects will be considered constructible from B that belong to the analytically defined adjunctive construction field. We shall call $V(A)$, $L(A)$, etc., *adjunctive operation sets*. To the distinction with this notion we shall call the sets V, L, etc., *adjunctionless operation sets*; the construction fields of the first kind, such as $G_V(B)$ etc., will be called *adjunctionless construction fields*.

Definition. *If B is such a basis in G that the construction fields $G_V(B)$ and $G_{V(A)}(B)$ with respect to the operation sets V and $V(A)$ are identical, while the point set contained in $G_V(B)$ includes B as a proper subset, B will be called a sufficient basis (with respect to V).*

Definition. Let Δ be a field, Δ' an extension field of Δ, and let ϑ be an element of Δ' When the question whether ϑ is algebraic or transcendent over Δ is left as it is, we shall call ϑ a *parameter* (over Δ). When the coordinates (τ_1, τ_2) of a point with respect to some coordinate system are parameters over a certain coordinate field Δ, they will be called *parametrical coordinates*.

The coordinates of a *geometrically* adjoined point will generally be denoted by parameters.

Finally we mention the following theorem.

Theorem 1. *Let be given a geometry G, an adjunctionless operation set V and a fixed field Δ. Let for any basis B that satisfies a certain condition C the following be true: representing the basis points by $z(0,0)$, $e_y(1,0)$, $t_\lambda(\alpha_\lambda, \beta_\lambda)$, $\lambda \in \Lambda$, Λ some index system (α_λ and β_λ being parameters over Δ, the coordinates taken with respect to some suitable coordinate system), a point $p(\pi_1, \pi_2)$ belongs to the construction field*

$G_V(B)$ *if and only if* π_1 *and* π_2 *are elements of the field* $\Delta(\alpha_\lambda, \beta_\lambda; \lambda \in \Lambda)$; *finally, let it have been proved that if a basis B satisfies the condition C, then also* $B' = [B, t]$ *arisen from B by the adjunction of a transcendental point t. Then we have:*

$G_{V(A)}(B) = G_V(B)$, i.e., *any object constructible from B with the aid of an adjunction operation* (but independent of the adjoined point, the adjunction operation considered a transcendental one) *is constructible without the use of an adjunction operation. So B is a sufficient basis.*

Demonstration. Let t (τ_1, τ_2) be a transcendental point over B; adjoin t to B, forming a new basis B'; according to the hypothesis a point p (π_1, π_2) belongs to the construction field $G_V(B')$ if and only if π_1, π_2 are elements of the field Δ $(\alpha_\lambda,\beta_\lambda,\tau_1,\tau_2; \lambda \in \Lambda)$; because p must belong to $G_{V(A)}(B)$ its coordinates must be independent of the coordinates of t. This means that the coordinates of p must be invariant when substituting another point transcendental over B for t. Consider one of the coordinates, say π_1, of a point p independent of the adjoined point t, with coordinates in $\Delta(\alpha_\lambda, \beta_\lambda, \tau_1, \tau_2)$; π_1 can be represented by a reduced polynomial fraction

$$\frac{\varphi(\tau_2)}{\psi(\tau_2)} = \frac{\mu_m \tau_2{}^m + \mu_{m-1} \tau_2{}^{m-1} + \qquad + \mu_0}{\nu_n \tau_2{}^n + \nu_{n-1} \tau_2{}^{n-1} + \qquad + \nu_0},$$

with coefficients in $\Delta(\alpha_\lambda, \beta_\lambda, \tau_1)$. Choose an integer k greater than m and n. Now t' $(\tau_1, \tau_2{}^k)$ is similarly transcendental over B. π_1 must be invariant when substituting $\tau_2{}^k$ for τ_2. We find: $\varphi(\tau_2)/\psi(\tau_2) = \varphi(\tau_2{}^k)/\psi(\tau_2{}^k)$, or after multiplication, $\varphi(\tau_2)\ \psi(\tau_2{}^k) = \varphi(\tau_2{}^k)\ \psi\ (\tau_2)$. This polynomial equation in τ_2 with coefficients in Δ $(\alpha_\lambda, \beta_\lambda, \tau_1)$ can be written: $\chi(\tau_2) + \mu_0(\nu_n \tau_2{}^n + \nu_{n-1} \tau_2{}^{n-1} + \ldots + \nu_0) - \nu_0 (\mu_m\tau_2{}^m + \mu_{m-1}\tau_2{}^{m-1} + \qquad + \mu_0)$, where $\chi(\tau_2)$ only contains powers of τ_2 greater than or equal to k.

Because τ_2 is transcendental over $\Delta(\alpha_\lambda, \beta_\lambda, \tau_1)$ all coefficients must vanish; this implies that $\varphi(\tau_2)/\psi(\tau_2)$ can be written μ_0/ν_0, μ_0 and ν_0 in Δ $(\alpha_\lambda, \beta_\lambda, \tau_1)$, or $\pi_1 \in \Delta(\alpha_\lambda, \beta_\lambda, \tau_1)$. In the same manner we can show: $\pi_1 \in \Delta(\alpha_\lambda,\beta_\gamma)$. Similarly we may find: $\pi_2 \in \Delta(\alpha_\lambda,\beta_\lambda)$.

Therefore p belongs to the adjunctionless construction field $G_V(B)$.

4. Automorphisms of construction fields.

Definition. *An automorphism of a figure F is a one-to-one mapping of*

F onto itself, points onto points, lines onto lines, circles onto circles, etc., all relations between the objects of F being invariant.

From the geometrical point of view automorphisms of finite figures will be considered; from the analytical point of view we shall deal with automorphisms of construction fields. In particular we are interested in the automorphisms of a construction field that leave a certain subset (e.g., the basis) objectwise fixed.

Examples.

1. Consider the Euclidean geometry E with the operation set V consisting of the ordinary active operations by ruler and compasses, the necessary decision operations, and the selection operations, as they will be described explicitly in Chapter VII. Let a basis B in E consist of two points, $z(0,0)$ and $e_y(1,0)$. Then reflection with respect to the line $L_1(z,\ e_y)$ (x-axis) represents an automorphism of the construction field $E_V(B)$ leaving B objectwise fixed.

2. Consider the projective geometry P with the objects: points and lines, with the relations: coincidence and incidence and their negations, and with the set L of operations: L_1 (producing the line through two distinct points), P_1 (producing the point of intersection of two distinct lines), D^{pp} (decision whether two points coincide or not), D^{ll} (decision whether two lines coincide or not), S_1 (selection operation). Let a basis B consist of four points, no three of which are on one line, $x(1,0,0)$, $y(0,1,0)$, $z(0,0,1)$, $e(1,1,1)$. A point p belongs to $P_L(B)$ if and only if the ratios of its homogeneous coordinates are elements of Π, field of the rational numbers (if $p = 0$). Let $t(\tau_1,\tau_2,1)$ be a transcendental point adjoined to B; so τ_1, τ_2 are transcendental elements over Π. Let $\mathfrak{a}$ be any automorphism of the field $\Pi(\tau_1,\tau_2)$ leaving Π fixed, mapping τ_1 onto $\mathfrak{a}(\tau_1)$, and τ_2 onto $\mathfrak{a}(\tau_2)$. Then the mapping $x \rightarrow x,\ y \rightarrow y,\ z \rightarrow z,\ e \rightarrow e,\ t \rightarrow t'\ (\mathfrak{a}(\tau_1),\ \mathfrak{a}(\tau_2),1)$ is an automorphism of $P_L(B')$ leaving $P_L(B)$ objectwise fixed, if $B' = [B,t]$.

5. Some remarks on the construction process.

A *construction problem* is the problem of constructing in a geometry G one or more figures H, or H_1, H_2, that satisfy certain prescribed conditions expressed in geometrical or in algebraical terms, whereby the construction must be carried out by the use of the operations of an adjunctionless operation set V and of the geometrical adjunction operation A^g, in a finite number of steps.

A series of operations producing a figure that satisfies the required conditions is called a *construction chain*. A construction chain is a certain solution of a certain construction problem. Because we can represent any construction operation by a symbol, a construction chain can be described by means of formulas compiled of these symbols.

Some problems are worded in such a manner that they have an infinite number of solutions: construction of *a* square, *an* isosceles right-angled triangle, construction of *a* triangle with given length of the base and given vertical angle; others are worded with respect to certain *data* in such a manner that they have only at most a finite number of solutions: given three non-concurrent lines, to construct *a* circle, or *the* circles, that touch these lines. The last problem has four solutions for the first requirement, and one for the second. Nevertheless, the same problem can be *sharpened, by adding certain conditions of order with respect to the same data,* so by using notions of the geometry itself and without adding more data, so that each of the circles can be described separately in his own manner.

Definition. When two construction problems A and B are worded with respect to the same set of data, A is called a *sharper* problem than B (or A is a *sharpening* of B), if each solution of A is a solution of B while there is at least one solution of B that is not a solution of A.

Definition. A construction problem worded with respect to a set of data in such a manner that it has at most a finite number of solutions, while no sharper problem is possible, it is called an *exact problem.*

Construction field theory is especially concerned with the possible solutions of exact construction problems.

The data of the problem will form the basis of a construction field. The construction of the required objects will then take place in the construction field, either $G_V(B)$ or $G_{V(A)}(B)$.

Let be given an exact construction problem in G: to construct a figure H in a construction field upon a given basis B.

Let a solution of the problem be represented by a construction chain K leading from B to H, K possibly containing a certain number of selection operations S_1.

Definition. If K does not contain any selection operation, or if H

is independent (not necessarily objectwise) of the selections made when performing the operations of K, H is called *constructible uniquely*. Then K will be called a construction chain with a *unique result*.

It will be proved that the figures constructible uniquely from a set B of data are those and only those that are invariant under all automorphisms of the construction field upon B that leave B objectwise fixed.

The set of those objects of G in the construction field $G_V(B)$ or $G_{V(A)}(B)$ that are constructible uniquely from B will be denoted by $G^u{}_V(B)$, or by $G^u{}_{V(A)}(B)$ respectively.

In the case that all active operations of V are univalent, we have appearently: $G^u{}_V(B) = G_V(B)$.

Examples. (Ordinary Euclidean geometry E.) (Fig. 1).

1. Given two distinct points a, b constituting a basis B; to construct *an* equilateral triangle a,b,c ($=$ *a* required figure H). Describe the circles $C_1(\dot{a},b)$ and $C_1(\dot{b},a)$ with the centres a and b, and passing through b and a respectively, and construct their points of intersection, c_1 and c_2.

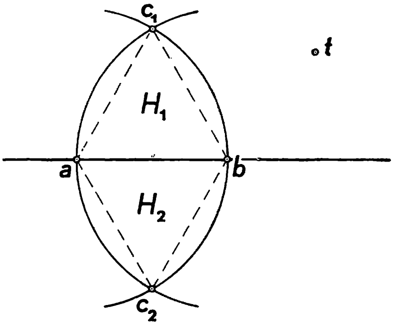

FIG. 1.

Let F be the figure formed by the points a, b and both circles. The points c_1 & c_2 are B-equivalent, even F-equivalent. Select one of the points c_1 & c_2: $S_1(c_1 \,\&\, c_2) = c$, and a triangle a,b,c is reached, solving the problem. Nevertheless, there are two possible solutions, H_1 and H_2. In Chapter VII it will be proved that the only automorphism (that leaves B objectwise fixed), but identity, of the construction field $E_V(B)$ is reflection with respect to the line

$L_1(a,b)$; so the automorphisms of $E_V(B)$ induce automorphisms of F that may be represented by the permutations $H_1 \leftrightarrows H_2$. So H is not constructible uniquely.

2. Given two distinct points a, b; to construct *a* rhomb a, c_1, b, c_2 with an angle of 120° in a (= the required figure H).

Describe $C_1(\dot{a},b)$ and $C_1(\dot{b},a)$; construct their points of intersection c_1 & c_2; select $S_1(c_1$ & $c_2) = c_1$ *and* c_2; a, c_1, b, c_2 is the required figure H. H is invariant (though not objectwise) with respect to the automorphisms of $E_V(B)$. H is constructible uniquely. It appears afterwards that the problem could have been worded: to construct *the* rhomb a, c_1, b, c_2 etc.

3. Given a basis B consisting of three non-collinear points a,b,t, while it is assumed known on which side of the line $L_1(a,b)$ point t is situated; to construct *an* equilateral triangle a,b,c, c being on the same side of $L_1(a,b)$ as t. The points c_1 and c_2 are F-equivalent, where F consists of the points a and b and the circles $C_1(\dot{a},b)$ and $C_1(\dot{b},a)$, but no longer B-equivalent. We can apply the $\gtrless$-decision operation $D^{p.p.l}$: It is possible to decide whether two points c and t, both excident with a line $\bar{c}$, are on the same side or on opposite sides of $\bar{c}$. Select $S_2(c_1$ & $c_2) = c$ *so that* c *is on the same side of* $L_1(a,b)$ *as* t. The required triangle is constructible uniquely. In Chapter VII it will be shown that the construction field $E_V(B)$ does not admit any automorphism leaving B objectwise fixed, but identity. Again it may be required to construct *the* equilateral triangle a,b,c etc.

4. Given a basis B consisting of three non-collinear points a,b,t, while the question on which side of the line $L_1(a,b)$ point t is situated is left as it is. This means that t is a *free point* over the figure formed by a, b, and the line $L_1(a,b)$; the construction field $E_V(B)$ may be considered a partially ordered construction field. $D^{p.p.l}$ cannot be applied to the points c_i, t and the line $L_1(a,b)$. The problem: to construct *an* equilateral triangle a,b,c, c being on the same side of $L_1(a,b)$ as t, is senseless, so *a fortiori* not solvable. Reflection with respect to $L_1(a,b)$ is again an automorphism of $E_V(B)$ that leaves B objectwise fixed.

CHAPTER II

LINEAR CONSTRUCTIONS IN PROJECTIVE PLANE

1. Objects, relations, operations.

In the ordinary twodimensional projective geometry P we have the objects: *points*, a, b, and *lines* $\bar{a}$, $\bar{b}$, ; in the present chapter we restrict ourselves to the relations of the first category (0-relations): *coincidence of two points*, written $a = b$ or $b = a$, *coincidence of two lines*, $\bar{a} = \bar{b}$ or $\bar{b} = \bar{a}$, and *incidence of a point and a line*, a inc $\bar{a}$ or $\bar{a}$ inc a; the negations of these relations (that will not be considered relations themselves) will be denoted by $a \neq b$ or $b \neq a$, $\bar{a} \neq \bar{b}$ or $\bar{b} \neq \bar{a}$, and a exc $\bar{a}$ or $\bar{a}$ exc a.

The axioms of the geometry are assumed to be well known. It is supposed that the geometry can be brought into a one-to-one correspondence with an analytical geometry over a field with some characteristic, in the usual manner.

We introduce the following construction postulates constituting the adjunctionless operation set L.

Active operations. (Univalent operations)

L_1: *There exists an operation* L_1, *which when applied to two distinct points* a *and* b, *produces the line* $L_1(a,b)$ *incident both with* a *and* b.

P_1: *There exists an operation* P_1, *which when applied to two distinct lines* $\bar{a}$ *and* $\bar{b}$, *produces the point* $P_1(\bar{a},\bar{b})$ *incident both with* $\bar{a}$ *and* $\bar{b}$.

Decision operations (0-decisions).

D^{pp}: *There exists an operation* D^{pp}, *which when applied to two points* a *and* b, *establishes whether* $a = b$ *or* $a \neq b$.

D^{ll}: *There exists an operation* D^{ll}, *which when applied to two lines* $\bar{a}$ *and* $\bar{b}$, *establishes whether* $\bar{a} = \bar{b}$ *or* $\bar{a} \neq \bar{b}$.

Selection operation.

S_1: *There exists an operation* S_1, *which when applied to a set of* (T-)*equivalent objects – given or constructed –, supplies a certain representative of this set.*

Explanations.

(i) Both L_1 and P_1 are *conditional* operations; their applicability depends on the possibility of the decision operations D^{pp} and D^{ll}. Consequently these operations are postulated on logical grounds, as motivated in the introduction.

(ii) We emphasize that we consider the points as *primary objects.* A line does not appear until it is constructed from two distinct points by means of the operation L_1. Our investigations concerning constructibilities, etc., will generally be restricted to points; the analogous questions with regard to lines may easily be reduced to those about points.

2. Construction theorem.

In the construction geometry (P, L) in the geometry P under the operations of the set L the *decision theorem* holds:

D^{pl}: *It is possible to decide whether a point a and a line $\bar{a}$ are incident or excident.*

Dem. Let us suppose $\bar{a}$ to have been constructed from two points p and q $(p \neq q)$. Apply D^{pp} to p and a; if $p = a$ then a inc $\bar{a}$; if $p \neq a$, then apply D^{ll} to $L_1(a,p)$ and $\bar{a}$. If these lines coincide, then a inc $\bar{a}$, and if not, then a exc $\bar{a}$.

3. Construction fields.

Let B be a basis of a construction field $P_L(B)$. We shall consider only those bases that consist of *points.*

If B contains four points, no three of which colline, we accomplish a one-to-one correspondence between the abstract points of B and a fundamental tetrad of a projective analytical geometry, using the available construction operations. By means of the selection operation S_1 and the decision operations D^{pp}, D^{ll}, and D^{pl} we select four fundamental points out of B which will be named x (1,0,0,) y(0,1,0), z(0,0,1), e(1,1,1). We consider the construction field $P_L(B)$, supposing the remaining basis points to bear parametrical coordinates $t_\lambda(\alpha_\lambda, \beta_\lambda, 1)$ or $(\gamma_\lambda, 1{,}0)$, $\lambda \in \Lambda$, Λ some index system. It is a well known theorem that a point of P belongs to $P_L(B)$ if and only if the ratios of its homogeneous coordinates are elements of the field $\Pi(\alpha_\lambda, \beta_\lambda, \gamma_\lambda; \lambda \in \Lambda)$, Π being the prime field with a certain characteristic contained in the field underlying our projective geometry.

The necessity is obvious, because the linear operations P_1 and L_1 are reflected by rational operations with regard to point coordinates and line equation coefficients. The sufficiency will follow from theorem 2, where it will be shown that there exist univalent construction chains for all points of $P_L(B)$ that satisfy the mentioned conditions.

The field $\Delta = \Pi(\alpha_\lambda, \beta_\lambda, \gamma_\lambda; \lambda \in \Lambda)$ is uniquely determined, apart from isomorphism. A projective coordinate transformation to another coordinate system with respect to fundamental points that belong to $P_L(B)$ itself, can be represented by a matrix composed of elements of $\Delta = \Pi(\alpha_\lambda, \beta_\lambda, \gamma_\lambda)$; so can its inverse. So Δ is independent of the selections made in order to fix a coordinate system. Δ is called *the field belonging to the construction field* $P_L(B)$.

4. Adjunction operations.

The use of the *geometrical adjunction operation* A^g (Cf. p. 12) will be admitted in order to procure the basis points for an adjunctionless construction field $P_L(B)$ and in order to construct objects in an adjunctive construction field $P_{L(A)}(B)$.

The definition of the latter may happen in an analogous way as in Chapter I, with a slight modification with regard to the fact that we consider homogeneous coordinates (Cf. p. 14).

(*a*) Let a construction field $P_L(B)$ upon a certain basis B be represented in analytical form, as in section 3. Let a point $t(\tau_1, \tau_2, 1)$ be adjoined to B by application of an adjunction operation. Consider the fields Δ, $\Delta(\tau_1)$, $\Delta(\tau_1, \tau_2)$. As we did in Chapter I we can define transcendental, semitranscendental, and algebraic adjunctions. As a consequence of theorem 1 (p. 16), that similarly may be modified with regard to the case of homogeneous coordinates, we find, that $P_L(B) = P_{L(A)}(B)$ in the case that B contains four points, no three of which colline. Consequently such a basis is *sufficient* (with respect to L).

The following bases are not sufficient:

(*b*) *B consisting of one point.* Both construction fields $P_L(B)$ and $P_{L(A)}(B)$ consist of that point only.

(*c*) *B consisting of two distinct points.* Both $P_L(B)$ and $P_{L(A)}(B)$ consist of these basis points and the line connecting them.

(*d*) *B consisting of three non-collinear points.* Both $P_L(B)$ and $P_{L(A)}(B)$ consist of these basis points and the lines connecting them.

(*e*) *B consisting of* (*at least three distinct*) *collinear points* $x(1,0,0)$, $y(0,1,0)$, $e_z(1,1,0)$, $t_\lambda(\alpha_\lambda, 1,0)$, $\lambda \in \Lambda$, *and of one point* $z(0,0,1)$ *excident with the line connecting them.* $P_L(B)$ consists of these points and the lines connecting them. The geometrical adjunction of a point t excident with all lines of $P_L(B)$ is a transcendental adjunction, because there can be found a coordinate system with respect to which t is transcendental over B. $B' = [B,t]$ is a sufficient basis, so that $P_{L(A)}(B)$ also contains the points p $(\pi,1,0)$, where π is an element of the field Π $(\alpha_\lambda;\ \lambda \in \Lambda)$. To construct these points the adjunction of a point e $(1,1,1)$ on the line $L_1(z,e_z)$ suffices. This adjunction is a semitranscendental one.

(*f*) *B consisting of* (*at least three distinct*) *collinear points,* $x(1,0,0)$, $y(0,1,0)$, $e_z(1,1,0)$, $t_\lambda(\alpha_\lambda,1,0)$, $\lambda \in \Lambda$. $P_L(B)$ consists of these points and the line connecting them. The geometrical adjunction of a point excident with this line is transcendental; after this adjunction we are back in the case (*e*); a second geometrical adjunction is transcendental again. We find: $P_{L(A)}(B)$ also contains the points $p(\pi,1,0)$, where π is an element of the field $\Pi(\alpha_\lambda)$. To construct these points the adjunction of a point $z(0,0,1)$ (transcendental) and the adjunction of a point e $(1,1,1)$ incident with $L_1(z,e_z)$ (semitranscendental) suffice. So both in (*e*) and in (*f*) we have: $P_L(B) \neq P_{L(A)}(B)$.

5. Formal description of the possible construction processes.

Theorem 2. *Let B be a basis in the projective geometry P; let r be a point, or let* $\bar{r}$ *be a line in the construction field* $P_L(B)$ *if B is a sufficient basis, and in the construction field* $P_{L(A)}(B)$ *if B is non-sufficient, then there exists a construction chain leading from B and having a unique result either r or* $\bar{r}$*, that can be expressed in formulas compiled of the symbols representing the operations of L, or L*(*A*).

Dem. We shall show that there exists a construction chain as described in the theorem for any of the *rational operations* with respect to the *reduced* homogeneous coördinates of the points, and to the *reduced* homogeneous equation coefficients of the lines in the adjunctionless construction field $P_L(B)$ upon a *sufficient* basis B. (The homogeneous coordinates of a point and the homogeneous equation coefficients of a line are called *reduced* if one of them amounts to 1.)

We restrict ourselves to the non-trivial cases mentioned in section 4, (*a*), (*e*), and (*f*). At first the bases of the last two cases will be extended to sufficient bases of construction fields in the manner as has already been shown in section 4; henceforth no more adjunction operations will be needed.

Next we suppose the basis B to be sufficient and, consequently, to contain four points, no three of which colline. Applying the selection operation S_1 and the available 0-decision operations a fundamental tetrad and a coordinate system is fixed: $x(1,0,0)$, $y(0,1,0)$, $z(0,0,1)$, $e(1,1,1)$. We call $L_1(x,y) = \bar{z}$, $L_1(x,z) = \bar{y}$, $L_1(y,z) = \bar{x}$. The following formulas are valid for any α and β, with only one exception (Case 3,(*d*)).

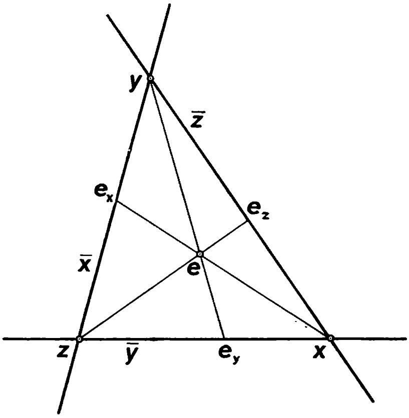

FIG. 2.

(1) *Isolation of a reduced homogeneous coordinate onto one of the three fundamental lines* $\bar{x}$, $\bar{y}$, *or* $\bar{z}$.

Two examples: let $p(\alpha, \beta, 1)$ be given, then $P_1\{L_1(p,y), \bar{y}\} = q(\alpha, 0, 1)$, and $P_1\{L_1(p,x), \bar{x}\} = r(0,\beta,1)$.

Similar isolation of a reduced homogeneous equation coefficient. Let $\bar{p}(\alpha x+\beta y+z = 0)$ be a given line, then $P_1(\bar{p},\bar{y}) = s(1,0,-\alpha)$, and $P_1(\bar{p},\bar{x}) = t(0,1,-\beta)$.

(2) *We pass from the points found under* (1) *to points on the line* $\bar{y}$, *with third coordinate* 1, using the construction chains:

(*a*) From $p(0,\alpha,1)$ to $q(\alpha,0,1)$:

$$P_1 < L_1\,[P_1\,\{L_1(p,x), L_1(z,e)\}, y], \bar{y} > = q.$$

(*b*) From $p(0,1,\alpha)$ to $q(\alpha,0,1)$:

$$P_1 < L_1\,[P_1\,\{L_1(p,x), L_1(y,e)\}, z], L_1(x,e) > = s(\alpha,1,1);$$

$$P_1\,\{L_1(s,y), \bar{y}\} = q.$$

(*c*) From $p(\alpha,1,0)$ to $q(\alpha,0,1)$:

$$P_1 < L_1 \, [P_1\{L_1(p,z), L_1(x,e)\}, y], \bar{y} > \, = q.$$

(*d*) From $p(1,\alpha,0)$ to $q(\alpha,0,1)$:

$$P_1 < L_1 \, [P_1 \{L_1(p,z), L_1(y,e)\}, x], L_1(z,e) > \, = s(\alpha,\alpha,1);$$
$$P_1 \{L_1(s,y), \bar{y}\} = q.$$

(*e*) From $p(1,0,\alpha)$ to $q(\alpha,0,1)$:

(*Notation*: $P_1 \{L_1(x,e), \bar{x}\} = e_x$, etc., cyclically.)

$$P_1 < L_1 \, [P_1 \{L_1(p,e_x), L_1(y,e)\}, e_z], \bar{y} > \, = q.$$

(3) *Construction chains for the rational operations with respect to the coordinates of points as found under* (2).

(*a*) From $p(\alpha,0,1)$ and $q(\beta,0,1)$ to $r(\alpha+\beta,0,1)$:

$$P_1 < L_1 \, [P_1\{L_1(p,y), L_1(x,e)\}, z], \bar{z} > \, = s(\alpha,1,0);$$
$$P_1 < L_1 \, [P_1 \{L_1(s,q), L_1(x,e)\}, y], \bar{y} > \, = r.$$

(*b*) From $p(\alpha,0,1)$, and $q(\beta,0,1)$ to $r(\alpha-\beta,0,1)$:

$$P_1 < L_1 \, [P_1 \{L_1(p,y), L_1(x,e)\}, q], \bar{z} > \, = s(\alpha-\beta,1,0);$$
$$P_1 < L_1 \, [P_1 \{L_1(s,z), L_1(x,e)\}; y], \bar{y} > \, = r.$$

(*c*) From $p(\alpha,0,1)$ and $q(\beta,0,1)$ to $r(\alpha\beta,0,1)$:

$$P_1 < L_1 \, [P_1 \{L_1(q,y), L_1(z,e)\}, x], \bar{x} > \, = s(0,\beta,1);$$
$$P_1 < L_1 \, [P_1 \{L_1(p,e_x), \bar{z}\}, s], \bar{y} > \, = r.$$

(*d*) From $p(\alpha,0,1)$ and $q(\beta,0,1)$ to $r(\alpha,0,\beta)$, if $\beta \neq 0$ (application of D^{pl} to q and $\bar{y}$):

$$P_1 < L_1 \, [P_1 \{L_1(q,y), L_1(x,e)\}, z], L_1(y,e) > \, = s(\beta,1,\beta);$$
$$P_1 < L_1 \, [P_1 \{L_1(p,y), L_1(x,e)\}, \; z], \; L_1(s,x) > \, = t(\alpha,1,\beta);$$
$$P_1 \{L_1(t,y), \bar{y}\} = r.$$

(4) *Reversely: from the coordinates to point or line.*

Two examples:

(*a*) From $p(\alpha,0,1)$ and $q(\beta,0,1)$ to the point $r(\alpha,\beta,1)$:

$$P_1 < L_1(p,y), L_1 \, [P_1 \{L_1(q,y), L_1(z,e)\}, x] > \, = r.$$

(*b*) From $p(\alpha,0,1)$ and $q(\beta,0,1)$ to the line $\bar{r}(\alpha x + \beta y + z = 0)$:

$$P_1 < L_1 \, [P_1 \{L_1(p,e_z), L_1(x,e)\}, e_y], L_1(z,e) > \, = r(1,1,1-\alpha);$$
$$P_1 < L_1 \, [P_1 \{L_1(q,e_z), L_1(x,e)\}, e_y], \bar{x} > \, = s(0,1,-\beta);$$
$$L_1 \, [P_1 \{L_1(r,e_x), \bar{y}\}, s] = \bar{r}.$$

So far the demonstration.

CHAPTER III

LINEAR CONSTRUCTIONS IN AFFINE PLANE

1. Objects, relations, operations.

When we consider affine geometry as a projective geometry in which a certain line has been appointed to play the part of the improper line, and when we assume the entire projective plane to be "attainable", as we did in Chapter II, it is clear that the operation of constructing a parallel to a given line, and the decision operation whether two lines are parallel or not, can be performed with the aid of the operations L_1, P_1, D^{pp}, D^{ll}, D^{pl} as mentioned in Chapter II.

In the following we consider the entire affine plane to be attainable *except the improper line.* We shall introduce new operations concerning parallelism. The operation P_1 of Chapter II will be replaced by P_1'

In affine geometry A we have the objects: points and lines, and the 0-relations: coincidence, incidence, parallelism. The axioms (incidence and parallelism) of affine geometry are assumed to be well known. It is supposed that the geometry can be brought into a one-to-one correspondence with an analytical geometry over a field with some characteristic, in the usual manner.

We introduce the following construction postulates constituting the adjunctionless operation set L.

Active operations. (Univalent operations)

L_1 (Chapter II, p. 22);

P_1': *There exists an operation* P_1', *which when applied to two distinct non-parallel lines* $\bar{a}$ *and* $\bar{b}$, *produces the point of intersection* $P_1'(\bar{a},\bar{b})$, *incident both with* $\bar{a}$ *and* $\bar{b}$.

L_2: *There exists an operation* L_2, *which when applied to a point* a *and a line* $\bar{a}$, a exc $\bar{a}$, *produces the line* L_2 (inc a, $//$ $\bar{a}$), *passing through* a *and parallel to* $\bar{a}$.

Decision operations.

D^{pp}, D^{ll} (Chapter II, p. 22).

Selection operation: S_1 (Chapter II, p. 22).

Remark. In the affine construction geometry (A, L) not every line will contain two distinct points *a priori*: the operation L_2 produces a line from one point only. Nevertheless, we can use an analogous proof as given in Chapter II, p. 23 for the **decision theorem:**

D^{pl}: *It is possible to decide whether a point a and a line $\bar{a}$ are incident or excident,*

because, in the proof in question, only one point of the line $\bar{a}$ is used.

Furthermore we have a second **decision theorem:**

D^{par}: *It is possible to decide whether two distinct lines are parallel or not.*

Dem. Let $\bar{a}$ and $\bar{b}$ be two lines of A, and let b be any point, b inc $\bar{b}$; then apply D^{ll} to $\bar{b}$ and $L_2(\text{inc b}, //\bar{a})$.

2. Construction fields.

We shall consider those bases for construction fields that consist of points only.

Let B be a given basis containing *three non-collinear points.* Using the selection operation S_1 and the available decision operations we select three non-collinear points out of B and name them $z(0,0)$, $e_y(1,0)$, and $e_x(0,1)$. The remaining basis points are denoted by parametrical coordinates $t_\lambda(\alpha_\lambda, \beta_\lambda)$, $\lambda \in \Lambda$. So B admits of a complete construction of an oblique cartesian coordinate system.

Consider the construction field $A_L(B)$. We assume it a well known fact that a point of A belongs to $A_L(B)$ if and only if its coordinates are elements of the field $\Pi(\alpha_\lambda, \beta_\lambda; \lambda \in \Lambda)$, Π being the prime field with a certain characteristic contained in the field underlying our affine geometry.

The necessity is obvious, because the linear operations L_1, L_2, and P_1' are reflected by rational operations with regard to point coordinates and line equation coefficients. The sufficiency will also follow from theorem 3, where it will be shown that there exist univalent construction chains for all points of $A_L(B)$ that satisfy the mentioned conditions.

The field $\Delta = \Pi(\alpha_\lambda, \beta_\lambda; \lambda \in \Delta)$ is uniquely determined, apart from isomorphism. An affine coordinate transformation to another coordinate system with respect to new points $z'(0,0)$, $e_y'(1,0)$, $e_x'(0,1)$ that belong to $A_L(B)$ itself, can be represented by a matrix composed of elements of Δ; so can its inverse. So Δ is independent of the selections made in order to fix a coordinate system. Δ is called *the field belonging to the construction field* $A_L(B)$. In this case $A_L(B)$ is called the *linear construction field upon B*.

3. Adjunction operations.

The use of the *geometrical adjunction operation* A^g (Cf. p. 12) will be admitted in order to procure the basis points for an adjunctionless construction field $A_L(B)$ and in order to construct objects in an adjunctive construction field $A_{L(A)}(B)$ (Cf. pp. 14, 15).

As a consequence of theorem 1 (p. 16) we find, that $A_L(B) = A_{L(A)}(B)$ in the case that B contains *three non-collinear points*. Consequently such a basis is *sufficient* (with respect to L).

The following bases are not sufficient:

(*a*) *B consisting of one point.* Both $A_L(B)$ and $A_{L(A)}(B)$ consist of that point only.

(*b*) *B consisting of a set of* (*at least two distinct*) *collinear points,* $z(0,0)$, $e_y(1,0)$, t_λ $(\alpha_\lambda,0)$, $\lambda \in \Lambda$. $A_L(B)$ consists of these points and the line connecting them. The geometrical adjunction of a point t excident with the line through the basis points is a transcendental adjunction, because there can be found a coordinate system with respect to which t is transcendental over B. $B' = [B,t]$ is a sufficient basis, so that $A_{L(A)}(B)$ also contains the points $p(\pi,0)$, where π is an element of $\Pi(\alpha_\lambda;\ \lambda \in \Lambda)$. So we have $A_L(B) \neq A_{L(A)}(B)$. Only one (transcendental) adjunction suffices to construct the objects of $A_{L(A)}(B)$.

In this case $A_{L(A)}(B)$ is called the *linear construction field* upon B.

4. Formal description of possible construction processes.

We shall give a series of construction chain formulas; they can be found by translation of the corresponding formulas valid in projective geometry into affine terminology.

Theorem 3. *Let B be a basis in affine geometry A; let r be a point, or let* $\bar{r}$ *be a line in the construction field* $A_L(B)$ *if B is sufficient, and in the construction field* $A_{L(A)}(B)$ *if B is non-sufficient, then there exists a construction chain leading from B and having a unique* *result either r or $\bar{r}$, that can be expressed in formulas compiled of the symbols representing the operations of L, or of $L(A)$.*

Dem. It suffices to show that there is a construction chain as described in the theorem for any of the *rational operations* with respect to the coordinates of the points, and to the *reduced* homogeneous equation coefficients of the lines in the adjunctionless construction field $A_L(B)$ upon a *sufficient* basis B.

At first a non-sufficient basis of the type mentioned in section 3, (*b*), will be extended to a sufficient basis in the manner mentioned above. Henceforth no more adjunction operations will be needed.

Next we suppose the basis B to be sufficient and, consequently, to contain three non-collinear points. Applying the selection operation S_1 and the available 0-decision operations a coordinate system is constructed: $z(0,0)$, $e_y(1,0)$, $e_z(0,1,)$. We call $L_1(z,e_y) = \bar{y}(y = 0)$; $L_1(z,e_x) = \bar{x}(x = 0)$; $P_1{}' \,\{L_2(\text{inc } e_y, /\!/\bar{x}), L_2(\text{inc } e_x, /\!/ \,\bar{y})\} = e(1,1)$.

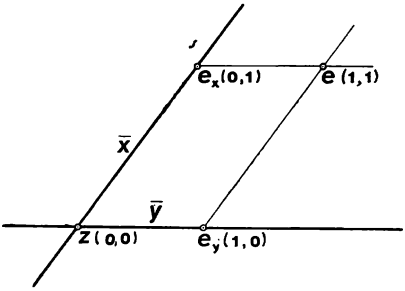

FIG. 3.

(1) *Isolation of a coordinate of a point onto the line $\bar{y}$.*

(*a*) From $p(\alpha,\beta)$ to $q(\alpha,0)$, $\alpha \neq 0$: $P_1{}' \,\{L_2(\text{inc}\, p, /\!/ \,\bar{x}), \bar{y}\} = q$.

(*b*) From $p(\alpha,\beta)$ to $q(\beta,0)$, $\beta \neq 0$:

$$P_1{}' < L_2 \,[\text{inc } P_1{}' \,\{L_2 \,(\text{inc}\, p, /\!/ \,\bar{y}), L_1(z,e)\}, /\!/ \,\bar{x}], \bar{y} > \; = q.$$

Isolation of the ratios of the equation coefficients of a line onto $\bar{y}$.

(*c*) From $\bar{p}(\alpha x + \beta y + \gamma = 0)$ to $q(-\gamma/\alpha,0)$, $\alpha \neq 0$: $P_1{}' \,(\bar{p},\bar{y}) = q$.

(*d*) From $\bar{p}(\alpha x + \beta y + \gamma = 0)$ to $q(\beta/\alpha,0)$, $\alpha \neq 0$:

if e_x inc $\bar{p}$: $P_1{}'(\bar{p}\bar{y},) = q$;

if e_x exc $\bar{p}$: $P_1{}' \,\{L_2(\text{inc } e_x, /\!/ \,\bar{p}),\bar{y}\} = q$.

(*e*) From $\bar{p}(\alpha x+\beta y+\gamma = 0)$ to $q(\gamma/\beta,0)$, $\beta \neq 0$:

$$P_1' \; [L_2 \{\text{inc } P_1'(\bar{p},\bar{x}), // \; \bar{y}\}, L_2 \{\text{inc } z, // \; L_1(e_x,e_y)\}] = r(\gamma/\beta,-\gamma/\beta);$$

$$P_1' \{L_2(\text{inc } r, // \; \bar{x}), \bar{y}\} = q.$$

(2) *The rational operations with respect to the abscissae.*

(*a*) From $p(\alpha,0)$ and $q(\beta,0)$ to $r(\alpha+\beta,0)$, $\alpha \neq 0$, $\beta \neq 0$:

$$L_1 \, [P_1' \{L_2(\text{inc } p, // \; \bar{x}), L_1(e_x,e)\}, z] = \bar{s}(x-\alpha y = 0);$$

$$P_1' < L_2 \, [\text{inc } P_1' \{L_2(\text{inc } q, // \; \bar{s}), L_1(e_x,e)\}, // \; \bar{x}], \bar{y} > \; = r.$$

(*b*) From $p(\alpha,0)$ and $q(\beta,0)$ to $r(\alpha-\beta,0)$, $\beta \neq 0$:

if $\alpha = 0$:

$$P_1' \, [L_2 \{\text{inc } z, // \, L_1(q,e_x)\}, L_1(e_x,e)] = s \, (-\beta,1);$$

if $\alpha \neq 0$:

$$L_1 \, [P_1' \{L_2(\text{inc } p, // \bar{x}), L_1(e_x,e)\}, q] = \bar{s};$$

$$P_1' \{L_2(\text{inc } z, // \; \bar{s}), L_1(e_x,e)\} = s(\alpha-\beta,1);$$

finally:

$$P_1' \{L_2(\text{inc } s, // \; \bar{x}), \bar{y}\} = r.$$

(*c*) From $p(\alpha,0)$ and $q(\beta,0)$ to $r \; (\alpha\beta,0)$, $\alpha \neq 0, \beta \neq 0$:

$$P_1' \{L_2(\text{inc } q, // \; \bar{x}), L_1(z,e)\} = s(\beta,\beta);$$

$$P_1' \{L_2(\text{inc } s, // \; \bar{y}), \bar{x}\} = t(0,\beta);$$

$$P_1' \{L_2\{\text{inc } t, // \, L_1(e_x,p)\}, \bar{y}] = r.$$

(*d*) From $p(\alpha, 0)$ and $q(\beta,0)$ to $r(\alpha/\beta,0)$, $\alpha \neq 0$, $\beta \neq 0$, $\beta \neq 1$: we construct the point $s(0,\beta)$ as we did in (*c*); then

$$P_1' \, [L_2 \{\text{inc } e_x, // \, L_1(s,p)\}, \bar{x}] = q.$$

(3) *From abscissae back to point and line.*

(*a*) From $p(\alpha,0)$ and $q(\beta,0)$ to $r(\alpha,\beta)$, $\beta \neq 0$: we construct the point $s(0,\beta)$ as we did in 2(*c*); then

$$P_1' \{L_2(\text{inc } p, // \; \bar{x}), L_2(\text{inc } s, // \; \bar{y})\} = r.$$

(*b*) From $p(\alpha,0)$ and $q(\beta,0)$ to $\bar{r}(\alpha x+\beta y+1 = 0)$, while not $\alpha = 0$ and $\beta = 0$ simultaneously:

$$\text{if } \alpha \neq 0 \text{ and } \beta \neq 0\text{: } P_1'[L_2\{\text{inc } e_y, // L_1(p,e_x)\}, \bar{x}] = s;$$

$$P_1' \, [L_2 \, (\text{inc } s, // \, L_1(z,e)\}, \bar{y}] = t(-1/\alpha,0);$$

$$P_1' < L_2 \, [\text{inc } P_1' \{L_1(p,e_x), \, L_1(e_y,e)\}, // L_1(z,e)], \, \bar{x} > \; = v(0,-1/\beta);$$

$$L_1 \, (t,v) = \bar{r}.$$

If $\alpha = 0$: the point t can be constructed first; then $L_2(\text{inc } t, // \bar{y}) = \bar{r}$;
If $\beta = 0$: the point s can be constructed; then $L_2(\text{inc } s, // \; \bar{x}) = \bar{r}$.

(*c*) From $p(\alpha,0)$ and $q(\beta,0)$ to $\bar{r}(\alpha x+\beta y = 0)$, $\alpha \neq 0$, $\beta \neq 0$: we construct the point $s(0,\alpha)$ according to 2(*c*) and substituting p for q; then $L_2\{\text{inc } z, // L_1(s,q)\} = \bar{r}$.

Consequently the solution of any solvable construction problem can be described by formulas compiled of the introduced operation symbols.

5. Construction theorem.

It will be convenient to introduce a symbol for the operation possible by virtue of the *active operation theorem* (only for characteristic $p \neq 2$):

M: *It is possible to construct the midpoint* $M(a,b) = M(b,a)$ *of a given point couple* a,b.

Dem. Apply D^{pp} to a and b; if $a = b$, we have $m = a$ too; now we assume that $a \neq b$.

Let the basis of a construction field $A_L(B)$ be sufficient. Then in $A_L(B)$ two points exist, say p and q, being collinear with a but excident with $L_1(a,b)$. Then we proceed:

$P_1'\,[L_2\{\text{inc}\ q,\ //\ L_1(a,b)\},\ L_1(p,b)] = r$; apply D^{par} to $L_1(a,r)$ and $L_1\ (b,q)$; if these lines are not parallel, then $P_1' < L_1\,[P_1'\{L_1(a,r),\ L_1(b,q)\},\ p],\ L_1(a,b) > \ = m$, being the point required. If the lines are parallel, then

$$P_1'\,[L_2\,\{\text{inc}\ p,\ //\ L_1(a,r)\},\ L_1(a,b)] = m.$$

When the basis is non-sufficient the performability of M requires one adjunction operation of a point p excident with $L_1(a,b)$.

6. Limited use of the operations concerning parallelism: $\mathbf{L_2}$ and $\mathbf{D^{par}}$.

It is a well known fact that given a *parallelogram* the affine linear constructions can be performed by using the ruler only; in our terminology this means, that the operation L_2 need be used only two times. We shall see that this wording is somewhat superficial; we shall deal with the theorems on this subject somewhat more carefully.

The usual auxiliary constructions to find arbitrary parallels are based on the same property that has been used above, when deriving the construction for the midpoint of a point couple. (Cf. fig. 4.) If $\bar{a}$ and $\bar{b}$ are parallel lines, then m is the midpoint of a,b (for characteristic $p \neq 2$), and conversely. It seems to us that this property is the simplest one to reach the purpose of constructing more parallels, requiring the least number of conditions for the given points and lines so that we may decide that the points of intersection of the occurring lines exist. So in the present section we shall always use this property. In order to avoid misunderstandings we shall use as careful expressions as possible, when stating the following theorems.

In the present section we shall assume that the operation set L will be extended to L' by adding the *decision postulate*

D^{par}: *There exists an operation* D^{par}, *which when applied to two distinct lines, establishes whether they are parallel or not;*

furthermore, it will be allowed, in the construction geometry (A, L'), to restrict the use of the active operation L_2 to at most one, two, or three times.

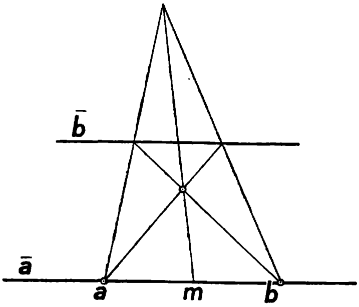

FIG. 4.

We shall start with bases that contain three non-collinear points, because such bases are sufficient in the construction geometry (A,L). In particular we prefer starting with bases that consist of exactly three non-collinear points. (For the remaining points of a basis may be considered as geometrically adjoined).

Because three non-collinear points are given, we cannot *a priori* say whether a certain geometrically adjoined point is algebraic or transcendental over the basis. We must speak geometrical language and only geometrical adjunctions are allowed. In order to describe the constructions in question it will be needed to use the available decision operations.

Theorem 4. *Let in the construction geometry* (A,L') *a figure* F *consist of three non-collinear points* a,b,c; *let it be allowed to use the active operation* L_2 *as a postulated one only for once, say to produce the line* $L_2\{ \text{ inc } c, \; // \, L_1(a,b)\} = \bar{c}$; *then it is possible to construct any parallel to* $L_1(a,b)$ *passing through an arbitrary point of* A *by using at least one* *and at most three geometrical adjunctions; if the characteristic p of A is not equal to 2, it seems necessary to use the decision operation D^{par} as a postulate, but if $p = 2$, this is no more needed.*

Dem. *The case $p \neq 2$.*

Only one improper point is determined bij the data.

Our first aim is the construction of the midpoint m of the point couple a, b. Adjoin (geometrically) a point d excident with the lines through a, b, and c, and excident with $\bar{c}$.
$P_1' \{L_1(a,d), \bar{c}\} = p$; $P_1' \{L_1(b,d), \bar{c}\} = q$; apply D^{par} to $L_1(a,q)$ and $L_1(b,p)$; if these lines are parallel, a second geometrical adjunction of a point e excident with all lines appeared hitherto, will follow; apply D^{par} to $L_1(d,e)$ and $\bar{c}$; if these lines are parallel, a third geometrical adjunction of a point f excident with all lines appeared hitherto, will follow. So we have found a point, that we shall call k, that is not the midpoint of the point couple c, $P_1' \{L_1(c,k), L_1(a,b)\}$ and that is excident both with $L_1(a,b)$ and $\bar{c}$.

$$P_1' \{L_1(a,k), \bar{c}\} = g;\ P_1' \{L_1(b,k), \bar{c}\} = h;$$

$$P_1' \{L_1(a,h), L_1(b,g)\} = s;\ P_1' \{L_1(k,s), L_1(a,b)\} = m.$$

Now let t be any point of A, t excident with $L_1(a,b)$ and with $\bar{c}$.
$P_1' \{L_1(t,a), \bar{c}\} = u$; apply D^{par} to $L_1(m,u)$ and $L_1(b,t)$; if these lines intersect, then $P_1' < L_1 [P_1'\{L_1(m,u), L_1(b,t)\}, a], L_1(b,u) > = v$; $L_1(t,v) = \bar{t}$, being the parallel required. If $L_1(m,u) \parallel L_1(b,t)$, we first construct a point u', $u' \neq u$, $u' \neq a$, u' incident with $L_1(t,a)$, and the construction of $\bar{t}$ runs according to the above formulas, after substituting u' for u.

The case $p = 2$.

Let t be any point of A, t excident with $L_1(a,b)$ and with $\bar{c}$.

$$P_1' \{L_1(t,a), \bar{c}\} = p;\ P_1' \{L_1(t,b), \bar{c}\} = q;$$

$L_1 [P_1' \{L_1(a,q), L_1(b,p)\}, t] = \bar{t}$, being the parallel required. There is made no use of D^{par}.

Theorem 5. *Let in the construction geometry (A,L') a figure F consist of three non-collinear points a, b, c; let it be allowed to use the active operation L_2 as a postulated one only twice, say to produce the lines $L_2 \{\text{inc } c, \parallel L_1(a,b)\} = \bar{c}$ and $L_2 \{\text{inc } b, \parallel L_1(a,c)\} = \bar{b}$; then it is possible to construct any parallel to a line in A and passing through a point of A by using exactly one geometrical adjunction; if the characteristic p of A is not equal to 2, it seems necessary to use the decision operation D^{par} as a postulate, but if $p = 2$, this is no more needed.*

Dem. *The case $p \neq 2$.*

Our first aim will be the construction of a figure that will be called a *parallel-triangle*. A *parallel-triangle* is the figure formed by three non-collinear points, the lines connecting them, and the three lines passing through the vertices parallel to the opposite sidelines. A parallel-triangle once obtained no more use will be made of the decision operation D^{par} as a postulate. To construct a parallel-triangle from the given *parallelogram* we shall use D^{par} as a postulate.

The point $P_1'(\bar{b}\bar{c})$, will be called d; $P_1'\ \{L_1(a,d),\ L_1(b,c)\} = s$. Adjoin (geometrically) a point g excident with any of the six lines connecting the points of the parallelogram. Apply D^{par} to $L_1(g,s)$ and $L_1(a,b)$. If these lines are parallel, then

$$P_1'\ \{L_1(g,s), L_1(a,c)\} = t;\ P_1'\ \{L_1(g,s), L_1(b,d)\} = u;$$

we find the parallel-triangle a,t,u, because $L_1(t,d)\ //\ L_1(a,u)$.

An analogously favourable situation is obtained when L_1 (g,s) is parallel to $L_1(a,c)$, a case which will not be dealt with in detail.

Now we suppose $L_1(g,s)$ to be parallel neither to $L_1(a,b)$ nor to $L_1(a,c)$.

$$P_1'\ \{L_1(g,s), L_1(a,c)\} = v;$$

$$P_1'\ < L_1\ [P_1'\ \{L_1(v,s),\ L_1(a,b)\},\ c],\ L_1(v,b)\ > \ = w;$$

then $L_1(v,w) = L_2\ \{\text{inc } a,\ //\ L_1(b,c)\}$, completing the construction of the parallel-triangle a,b,c.

Henceforth we continue the proof under the assumption that a parallel-triangle is available. The decisions concerning parallelism can be made by proof in the following.

We suppose the elements of the parallel-triangle to bear the names: the vertices f_1, f_2, f_3; the sidelines $L_1\ (f_\mu, f_\nu) = \bar{f}_\lambda$ the parallels $L_2\ (\text{inc } f_\lambda,\ //\ \bar{f}_\lambda) = \bar{e}_\lambda$; the opposite points $P_1'\ (\bar{e}_\mu, \bar{e}_\nu) = e_\lambda$, $\lambda,\mu,\nu = 1,2,3(\lambda \neq \mu \neq \nu \neq \lambda)$. (Figure 5).

We give a complete series of construction chains for parallels in the various cases.

(i) $L_2(\text{inc } e_\lambda,\ //\ \bar{f}_\lambda)$.

$L_1\{P_1' < L_1\ [P_1'\ \{L_1(e_\lambda,f_\lambda),\ \bar{f}_\lambda\},\ e_\nu],\ \bar{f}_\mu >,\ e_\lambda\}$ is the parallel required.

(ii) $L_2(\mathit{inc}\ a,\ //\ \bar{f}_\mu)$, a *inc* $\bar{f}_\nu$, $a \neq f_\lambda$, $a \neq f_\mu$.

According to (i) we construct $L_2(\text{inc } e_\nu)\ //\ \bar{f}_\nu = \bar{m}$; next

$$P_1'\ [L_1\ \{P_1'(\bar{f}_\lambda,\bar{m}),\ e_\lambda\},\ \bar{f}_\mu] = b;$$

$P_1'\ < L_1\ [P_1'\ \{L_1(f_\nu,a),\ \bar{m}\},\ b],\ \bar{e}_\nu > \ = c$; $L_1(a,c)$ is the parallel required.

(iii) L_2(*inc a*, $//\, \bar{f}_\mu$), *a inc* $\bar{e}_\nu$, $a \neq f_\nu$, $a \neq e_\lambda$, $a \neq e_\mu$.

Since L_2(inc e_λ, $//\, \bar{f}_\lambda$) is constructible (*i*), we may consider the points e_λ, f_μ, f_ν as a new parallel triangle. The problem has been reduced to that of (*ii*).

(iv) L_2(*inc a*, $//\, \bar{f}_\mu$), *a being excident with any of the six main lines of the parallel-triangle.*

We pass from the parallel-triangle f_λ, f_μ, f_ν to another one f_λ', f_μ, f_ν' by an affine transformation:
$P_1' \{L_1(a,f_\mu), \bar{f}_\mu\} = f_\lambda'$; $P_1' \{L_1(e_\nu, f_\lambda'), L_2(\text{inc } e_\mu, //\, \bar{f}_\mu)\} = e_\mu'$, according to (i); $P_1' \{L_1(e_\lambda,e_\mu'), \bar{f}_\mu\} = f_\nu'$

The problem has been reduced to that of (ii).

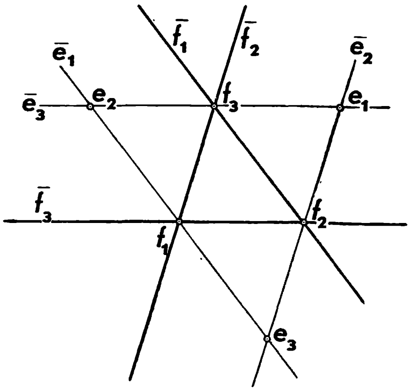

FIG. 5. Parallel-triangle

(v) L_2(*inc a*, $//\, \bar{a}$), *a being excident with any of the six main lines of the parallel-triangle, $\bar{a}$ being not parallel to any of these main lines.*

We remark that no decision *postulate* D^{par} is needed here, because a parallel to any of the main lines can be *constructed*; so we can *prove* whether a line is parallel to any of the main lines or not.

We pass to another parallel-triangle f_λ'', f_μ'', f_ν'' by an affine transformation, the axis of which will be $L_1 \{P_1'(\bar{a},\bar{e}_\lambda), e_\lambda\} = \bar{n}$.
$P_1'(\bar{a},\bar{e}_\nu) = e_\mu''$; $P_1'(\bar{a},\bar{f}_\nu) = f_\lambda''$; $P_1' \{\bar{a}, L_2(\text{inc } e_\nu, //\, \bar{e}_\nu)\} = e_\nu''$;
$P_1' \{L_1(e_\lambda,e_\nu''), \bar{f}_\nu\} = f_\mu''$; $P_1' [L_1 \{P_1'(\bar{a},\bar{f}_\lambda), f_\mu''\}, \bar{e}_\nu] = f_\nu''$

Thus the problem has been reduced to that of (iv).

The case $p = 2$.

The figure formed by the points *a*, *b*, *c*, and $P_1'(\bar{b},\bar{c}) = p$ is a parallelogram, but *a,b,c* constitute already a parallel-triangle, because $L_1(a,p)\ //\ L_1(b,c)$.

The construction of a parallel to any of the six main lines is contained in the result of theorem 4.

The construction of a parallel to an arbitrary line being not parallel to any of the main lines is easily reduced to the former case.

Let $\bar{t}$ be a given line. We pass to another parallelogram (= parallel-triangle) in which $\bar{t}$ becomes a main line.

The given points are a, b, c (non-collinear); according to the assumption of the theorem we may construct the parallels $L_2\{\text{inc } b, // L_1(a,c)\} = \bar{b}$ and $L_2\{\text{inc } c, // L_1(a,b)\} = \bar{c}$; $P_1'(\bar{b},\bar{c}) = p$.

$$P_1'\{L_1(a,c), \bar{t}\} = f;\ P_1'[L_2\{\text{inc } f, // L_1(a,b)\}, \bar{b}] = h;$$
$$P_1'(\bar{t},\bar{b}) = g;\ P_1'[L_2\{\text{inc } g, // L_1(a,b)\}, L_1(a,c)] = k.$$

Then f,h,g,k is the parallelogram (parallel-triangle) required which enables us to construct parallels to $\bar{t}$ (main line) without the use of the decision postulate D^{par}, proving the theorem.

As a corollary of the demonstration of theorem 5 we have in an affine construction geometry with characteristic $p \neq 2$:

Theorem 6. *Let a, b, c be three non-collinear points in the construction geometry (A, L'); let it be allowed to use the active operation L_2 at least three times as a postulated operation, say to produce the lines $L_2\{\text{inc } c, // L_1(a,b)\}$, $L_2\{\text{inc } b, // L_1(a,c)\}$, and $L_2\{\text{inc } a, // L_1(b,c)\}$, then it is possible to construct any parallel in A without any further use of L_2 and of the decision operation D^{par} as postulated operations, and of any adjunction operation.*

Remarks 1. In the geometry A with characteristic $p \neq 2$ the parallel-triangle appearently contains more constructive and geometrical properties than a parallelogram does; the fact that a parallelogram is given does not exclude the necessity of the use of the decision postulate D^{par}, but when a parallel-triangle is given, all decisions concerning parallelism can be made by proof; it is not known to us what may be the connection of this constructive superiority of the parallel-triangle over the parallelogram with the geometrical difference between both figures. Perhaps there is some indication in the fact that the parallel-triangle f_1, f_2, f_3 with the three opposite points $P_1'(\bar{e}_\mu,\bar{e}_\nu) = e_\lambda$, extended with the improper line and the three lines $L_1(f_\lambda,e_\lambda)$, embodies DESARGUES' configuration.

2. With respect to the construction fields upon a basis B containing three non-collinear points we remark:

(*a*) If it is allowed to use the active operation L_2 as a postulate at least three times, we can apply it in such a manner that the adjunctionless construction field $A_L'(B)$ is the *linear construction field upon* B, i.e., $A_L'(B) = A_L(B)$.

(*b*) If it is allowed to use L_2 as a postulate only twice, we can apply it in such a manner, that either the adjunctionless construction field $A_L'(B)$ or the adjunctive construction field $A_{L'(A)}(B)$ is the *linear construction field.*

(*c*) If it is allowed to use L_2 as a postulate only once, the construction fields both $A_L'(B)$ and $A_{L'(A)}(B)$ are proper subsets of the linear construction field. For, other parallels than those parallel to the first couple of parallel lines are not defined; only one improper point has been determined. The structure of these construction fields will not be investigated.

CHAPTER IV

RULER CONSTRUCTIONS AND UNIVALENT CONSTRUCTIONS WITH COMPASSES IN REFLECTION GEOMETRY

1. Objects, relations, operations.

A geometrical system satisfying the axioms on *incidence* and *parallelism*, and such a set of axioms on *congruence*, that they are significant in an analytical geometry over a field Δ with characteristic $p \neq 2$ and the property: -1 is not a square, is called a *reflection geometry*. The name is chosen in connection with the fact that reflections with respect to a point and to a line are possible in this geometry and play a rather important part in the constructions.

We shall not set up a sufficient set of congruence axioms; they may be derived from an analytically defined reflection geometry; concerning this analytical definition, that may take place with respect to a coordinate system called "the absolute coordinate system", we make some remarks. Points and lines may be defined in the usual manner, so may parallelism. Congruence may be defined for point couples and line couples; distance with respect to the distance of the "absolute" points (0,0) and (1,0) may be introduced as a *function* denoted by dis^2; $\mathrm{dis}^2(a,b)$ is the function $(\alpha_1-\beta_1)^2 + (\alpha_2-\beta_2)^2$, if (α_1,α_2) and (β_1, β_2) are the absolute coordinates of a and b respectively. This leads to the objects: circles, with radius $\neq 0$. Given two lines $\bar{a}(\alpha_1 x+\alpha_2 y+\alpha_3 = 0)$ and $\bar{b}(\beta_1 x+\beta_2 y+\beta_3 = 0)$, the angle $\widehat{\bar{a}\bar{b}}$ may be introduced as a function $\cos^2$: $\cos^2(\bar{a}, \bar{b}) = (\alpha_1\beta_1+\alpha_2\beta_2)^2 \; (\alpha_1{}^2+\alpha_2{}^2)^{-1} \; (\beta_1{}^2+\beta_2{}^2)^{-1}$.

Without proof we mention the following property of plane reflection geometry:

A coordinate transformation (similarity transformation) from a coordinate system S to another one, S', based on two points s and t, having the coordinates (σ_1, σ_2) and (τ_1, τ_2) respectively with respect to S, and getting coordinates (0,0) and (1,0) respectively with respect to S', can only be defined in the case $\mathrm{dis}^2(s,t) \neq 0$. The transformation formulas read, calling $\pi = \mathrm{dis}^2(s,t) = (\tau_1-\sigma_1)^2+(\tau_2-\sigma_2)^2$:

$x' = \{(\tau_1-\sigma_1)\ x + (\tau_2-\sigma_2)\ y + \sigma_1{}^2+\sigma_2{}^2-\sigma_1\tau_1-\sigma_2\tau_2\}\ \pi^{-1},$
$y' = \{-(\tau_2-\sigma_2)\ x+(\tau_1-\sigma_1)\ y-\tau_1\sigma_2+\tau_2\sigma_1\}.\ \pi^{-1}$ or
$y' = \{(\tau_2-\sigma_2)\ x-(\tau_1-\sigma_1)\ y + \tau_1\sigma_2-\tau_2\sigma_1\}.\ \pi^{-1}.$

(The transformation is not uniquely determined by the data.)

In the present construction geometry the elementary objects: points, lines, and circles, will be considered. Relations: coincidence, incidence, parallelism, normality, congruence.

The adjunctionless operation set L of postulated operations will be introduced:

Active operations. (Univalent operations.)

L_1 (Chapter II, p. 22); P_1' (Chapter III, p. 28);

Refl $_b$ (a): *There exists an operation Refl, which when applied to an ordered couple of points a, b, produces the reflection point Refl b (a) of a with respect to b.*

Refl $_{\bar{a}}$ (a): *There exists an operation Refl, which when applied to a point a and a line $\bar{a}$, produces the reflection point Refl $_{\bar{a}}$ (a) of a with respect to $\bar{a}$.*

C_1: *There exists an operation C_1, which when applied to an ordered couple of distinct points a,b, produces the circle $C_1(\dot{a},b)$, with centre a and with circumference point b.*

Decision operations (0-decisions): D^{pp}, D^{ll} (Chapter II, p. 22).

Selection operation: S_1 (Chapter II, p. 22).

2. Construction theorems.

In the construction geometry (R,L) we have the following construction theorems:

D^{pl}: *It is possible to decide whether a point a and a line $\bar{a}$ are incident or excident.*

Dem. As in Chapter II, p. 23.

L_2: *It is possible to construct the line passing through a point a and parallel to a line $\bar{a}$, a exc $\bar{a}$.*

Dem. $\bar{a}$ having been constructed from two distinct points a_1 and a_2 by using the operation L_1, we continue:

Refl $_{a_1}$ $(a) = b$; *Refl* $_{a_2}(b) = c$; than $L_1(a,c)$ is the parallel required.

D^{par}: *It is possible to decide whether two distinct lines are parallel or not.*

Dem. Since any parallel can be *constructed* according to the performability of the active operation L_2 without the use of any decision operation concerning parallelism, the decision in question can always be made by a proof. For let $\bar{a}$ and $\bar{b}$ be two given distinct lines, and let a_1 be any point of $\bar{a}$, then apply D^{ll} to $\bar{a}$ and to L_2 (inc a_1, $//$ $\bar{b}$).

L_3: *It is possible to construct a line passing through a given point* a, *and normal to a given line* $\bar{a}$, *written* L_3 (*inc* a, $\perp$ $\bar{a}$).

Dem. If a exc $\bar{a}$, then: $\text{Refl}_{\bar{a}}(a) = b$, and $L_1(a,b)$ is the normal required.

If a inc $\bar{a}$, *we assume that there is given a point* a' exc $\bar{a}$; then L_3 (inc a', $\perp$ $\bar{a}$) $= \bar{b}$; if a inc $\bar{b}$, then $\bar{b}$ is the normal required, and if a exc $\bar{b}$, L_2 (inc a, $//$ $\bar{b}$) is the normal required.

D^{norm}: *It is possible to decide whether two lines are normal or not.*

Dem. Since any normal can be *constructed* according to the performability of the active operation L_3, the decision in question can always be made by a proof. For let $\bar{a}$ and $\bar{b}$ be two given distinct lines and let a_1 be any point of $\bar{a}$, then apply D^{ll} to $\bar{a}$ and to L_3(inc a_1, $\perp$ $\bar{b}$).

D^{pc}: *It is possible to decide whether a point* a *and a circle* $\bar{\bar{a}}$ *are incident or excident.*

Dem. Any circle in R only occurs when constructed from centre and circumference point, by the aid of an active operation. Circle $\bar{\bar{a}}$ is assumed to have been constructed from centre c and circumference point d, so $\bar{\bar{a}} = C_1(c,d)$. Apply D^{pp} to a and c, and to a and d. If $a = c$, then a exc $\bar{\bar{a}}$; if $a = d$, then a inc $\bar{\bar{a}}$. If $a \neq c$, and $a \neq d$, we operate: $L_3\{\text{inc } c, \perp L_1(a,d)\} = \bar{c}$; $\text{Refl}_{\bar{c}}(d) = e$; apply D^{pp} to a and e; if $a = e$, then a inc $\bar{\bar{a}}$, and if $a \neq e$, then a exc $\bar{\bar{a}}$.

D^{cc}: *It is possible to decide whether two circles* $\bar{\bar{a}}$ *and* $\bar{\bar{b}}$ *coincide or not.*

Dem. Both $\bar{\bar{a}}$ and $\bar{\bar{b}}$ have been constructed from centres and circumference points. Apply D^{pp} to their centres, and D^{pc} to $\bar{\bar{a}}$ and the circumference point of $\bar{\bar{b}}$.

P_2: *Given a line* $\bar{a}$ *and a circle* $\bar{\bar{a}}$, *both incident with a same point* p, *then it is possible to construct the "other" point of intersection of* $\bar{a}$ *and* $\bar{\bar{a}}$ P_2 $\{\bar{a}$ (*inc* a), $\bar{\bar{a}}$ (*inc* a)$\}$ *or* P_2 $(\bar{a},\bar{\bar{a}})$.

Dem. Let a be the centre of $\bar{\bar{a}}$; $L_3(\text{inc } a, \perp \bar{a}) = \bar{b}$; then $Refl_{\bar{b}}(p)$ is the point required.

P_3: *Given two distinct circles $\bar{\bar{a}}$ and $\bar{\bar{b}}$, both incident with a same point p, then it is possible to construct the "other" point of intersection of the circles, $P_3\{\bar{\bar{a}}(\text{inc } p), \bar{\bar{b}}\ (\text{inc } p)\}$ or $P_3\ (\bar{\bar{a}},\bar{\bar{b}})$.*

Dem. Let a be the centre of $\bar{\bar{a}}$, b the centre of $\bar{\bar{b}}$; $L_1(a,b) = \bar{c}$; then $\mathrm{Refl}_{\bar{c}}(p)$ is the point required.

C_2: *It is possible to construct a circle with a prescribed centre m and a prescribed radius $\widehat{\varrho} = \widehat{ab} \neq 0$.* Notation: $C_2(\dot{m},\widehat{ab})$.

Dem. Let a,b,m be three non-collinear points. Then

$$C_1 < \dot{m},\ P_1'\ [L_2\{\text{inc } a, // L_1(b,m)\}, L_2\{\text{inc } m, // L_1(a,b)\}] >$$

is the circle required.

If a,b,m colline, the construction of $C_2(\dot{m},\widehat{ab})$ requires a preceding auxiliary construction, which will not be described.

For the sake of uniformity we give a corresponding notation for the operation C_1, producing a circle with prescribed centre a and circumference point b: $C_1(\dot{a}, \widehat{ab})$.

3. Construction fields.

Consider a basis B for a construction field $R_L(B)$. If B is the empty basis, or if B consists of only one point, the structure of $R_L(B)$ is not interesting. First we consider the case of a basis consisting of a set of (at least two distinct) collinear points.

Theorem 7. *When $z(0,0)$, $e_y(1,0)$, $t_\lambda(\alpha_\lambda,0)$, $\lambda \in \Lambda$* (Λ some index system), *constitute a basis B of a construction field $R_L\ (B)$, a point $p(\pi,0)$ belongs to $R_L(B)$ if and only if π belongs to the class of finite linear expressions $\nu_o + \nu_1\alpha_1 + \nu_2\alpha_2 + \quad$, the coefficients ν_o being even integers with at most one exception, which may be an odd integer.*

Dem. (1) *Necessity.*

The basis points satisfy the condition of the theorem. We may say that the reflection with respect to a point is the only process occurring effectively. Let $a(\alpha,0)$ and $b(\beta,0)$ be two points satisfying the condition of the theorem, then the point $Refl_b(a) = c(2\beta-\alpha,0)$ also satisfies it.

(2) *Sufficiency.*

We will give a construction chain from B to the point $p(\pi,0)$, where π is some finite linear expression $\nu_o + \nu_1\alpha_1 + \nu_2\alpha_2 + \quad$ with coefficients ν_σ being even integers except perhaps one, say ν_0 without loss of generality. We mention the following constructions with respect to some point $t(a,0)$: $Refl_t(z) = t^{(2)}\ (2a,0)$; $Refl_{t^{(\nu)}}\ (t^{(\nu-1)}) = t^{(\nu+1)}\ ((\nu+1)a,0)$; $Refl_z(t^{(\nu)}) = t^{(-\nu)}\ (-\nu a,0)$.

Consequently, given a certain fixed integer ν the construction of the point $t^{(\nu)}\ (\nu a,0)$ can be performed unambiguously. So in our problem we may construct all points $t_0{}^{(\nu_0)}\ (\nu_0,0)$, $t_\lambda{}^{(\frac{1}{2}\nu_\lambda)}\ (\frac{1}{2}\nu_\lambda a_\lambda,0)$, $\lambda \in \Lambda$. Assuming these points available we continue:

$$Refl_{t_1}(\tfrac{1}{2}\nu_1)\,\{Refl_z(t_0{}^{(\nu_0)})\} = p_1(\nu_0+\nu_1 a_1,0),$$
$$Refl_{t2}\,(\tfrac{1}{2}\nu_2)\,\{Refl_z(p_1)\} = p_2\,(\nu_0+\nu_1 a_1+\nu_2 a_2,0),$$
, etc.

After a finite number of steps point $p(\pi,0)$ will be obtained, so p belongs to $R_L(B)$, completing the demonstration.

The construction field $R_L(B)$ upon a basis B containing three non-collinear points will be called the **orthogonal-linear construction field upon B.** We shall see that there are two types of orthogonal-linear construction fields according to the special structure of the basis. To explain this we shall define the concepts of P-field and Γ-field.

Let the points $z(0,0)$, and $e_y(1,0)$ be selected from B; the remaining basis points $t_\lambda\ (a_\lambda,\beta_\lambda)$, $\lambda \in \Lambda$, Λ some index system, will have parametrical coordinates $(a_\lambda,\beta_\lambda)$ with respect to one of the two possible orthogonal cartesian coordinate systems upon z and e_y. (Consequently, with respect to the other one, the points t_λ would have coordinates $(a_\lambda,-\beta_\lambda)$.)

Let Π be the prime field of a certain characteristic $p \neq 2$, contained in the field underlying our reflection geometry.

The field $\Pi(a_\lambda;\ \lambda \in \Lambda)$, Λ some index system, will be called Δ; the field $\Delta(\beta_\lambda)$ will be called P. A point is called a P-point if its coordinates (with respect to the coordinate system in question) are elements of P; a line is called a P-line, and a circle is called a P-circle, if the ratios of their equation coefficients belong to P; *the set of all P-objects is called the P-field* (upon the basis B consisting of $z(0,0)$, $e_y(1,0)$, $t_\lambda(a_\lambda,\beta_\lambda)$).

Furthermore: the P-points $p(\pi_1,\pi_2)$, such that π_1 is an *even rational expression in the* β_λ *together, with coefficients in* Δ (shortly: an even expression in the β_λ) and π_2 is an *odd expression in the* β_λ, are called Γ-points; the P-lines $r(y = \varrho_1 x+\varrho_2)$, such that ϱ_1 and ϱ_2 are odd expressions in the β_λ, are called Γ-lines; so are the P-lines

$\bar{s}(x = \sigma)$, such that σ is an even expression in the β_λ. The P-circles $\bar{\bar{t}}(x^2+y^2+\tau_1 x+\tau_2 y+\tau_3 = 0)$, such that τ_1 and τ_3 are even expressions and τ_2 is an odd expression in the β_λ, are called Γ-circles. *The set of all Γ-objects is called the* Γ-field (upon the basis B).

Remark. A better name for Γ-field is: the Γ-field with respect to the subfield $\Pi(\alpha_\lambda;\ \lambda \in \Lambda)$ of P.

If it is clear from the context what is meant, we shall only use the mere word Γ-field. Only in the demonstration of theorem 19 we shall, for the sake of clearness, add the supplement mentioned.

Theorem 8. *Both P-field and Γ-field are constructively closed with respect to the active operations of L.*

Dem. The line passing through two P-points (Γ-points) is a P-line (Γ-line) (operation L_1); the point of intersection of two P-lines (Γ-lines) is a P-point (Γ-point) (operation P_1'); a circle with P-centre (Γ-centre) and incident with a P-point (Γ-point) is a P-circle (Γ-circle) (operation C_1), as a simple calculation may show.

A similar calculation shows, that a reflection point of a P-point (Γ-point) with respect to a P-point (Γ-point) or a P-line (Γ-line) is a P-point (Γ-point) again.

Theorem 9. *When $z(0,0)$, $e_y(1,0)$, $t_\lambda(\alpha_\lambda,\beta_\lambda)$, $\lambda \in \Lambda$, constitute a basis B in the construction geometry (R,L), the orthogonal-linear construction field $R_L(B)$ is either the Γ-field or the P-field upon B. It is the P-field if and only if there exists an algebraic relation between the β_λ that can be written $\varphi'(\beta_\lambda) = \varphi''(\beta_\lambda) \neq 0$, where φ' is an even and φ'' is an odd rational expression in the β_λ with coefficients in $\Delta = \Pi(\alpha_\lambda;\ \lambda \in \Lambda)$.*

Dem. We remind the reader that the operations L_2 and L_3 producing respectively parallels and normals, are available. We shall show, that on the x-axis any point $p_1(\pi_1, 0)$ can be constructed if and only if π_1 is an even expression in the β_λ, and that on the y-axis any point $p_2(0,\pi_2)$ is constructible if and only if π_2 is an odd expression in the β_λ. So the orthogonal-linear construction field is, in any case, at least a Γ-field.

Let us assume this to have been proved for a moment. Then the second statement of the theorem can be proved as follows.

Given the existence of an algebraic relation between the β_λ, $\varphi'(\beta_\lambda) = \varphi''(\beta_\lambda) \neq 0$, where φ' is an even and φ'' is an odd rational expression of the β_λ. The points $k_1(\varphi', 0)$ on the x-axis and $k_2(0, \varphi'')$

on the y-axis are constructible according to the assumption; then

$$P_1' \, [L_2\{\text{inc } e_y, // L_1(k_1, k_2)\}, L_3\{\text{inc } z, \perp L_1(z, e_y)\}] = e_x(0,1);$$

consequently the entire *linear construction field* upon the basis $B' = [B, e_x]$ is attainable (Cf. affine geometry); but this is the P-field upon B.

Conversely, if the orthogonal-linear construction field is a P-field, it contains the point $e_x(0,1)$. Now e_x is a point on the y-axis and consequently 1 must be an odd rational expression in the β_λ with coefficients in Δ. But 1 is evidently an even expression in the β_λ so we find an algebraic relation as required.

Now we prove the first statement of the theorem.

(1) *Necessity.*

The basis point $z(0,0)$, $e_y(1,0)$, $t_\lambda(\alpha_\lambda, \beta_\lambda)$, $\lambda \in \Lambda$, (constituting the basis B) are points of the Γ-field upon B, because the abscissae are even and the ordinates are odd rational expression in the β_λ with coefficients in Δ; the Γ-field is constructively closed with respect to the operations of L, and consequently a point $p_1(\pi_1, 0)$ on the x-axis can only be constructed if π_1 is an even expression in the β_λ, and a point $p_2(0, \pi_2)$ on the y-axis can only be constructed if π_2 is an odd expression in the β_λ.

(2) *Sufficiency.*

First we construct some points on the x-axis, the abscissae of which are certain even expressions in the β_λ.

$$L_1(z, e_y) = \bar{y};\ L_3(\text{inc } z, \perp \bar{y}) = \bar{x};$$

$$P_1' \, \{L_2(\text{inc } t_\lambda, // \bar{x}), \bar{y}\} = k_\lambda(\alpha_\lambda, 0);\ P_1' \, \{L_2(\text{inc } t_\lambda, // \bar{y}), \bar{x}\} = m_\lambda(0, \beta_\lambda);$$

$$P_1' \, \{L_2(\text{inc } t_\lambda, // \bar{y}), L_2(\text{inc } e_y, // \bar{x})\} = n_\lambda(1, \beta_\lambda);$$

$$P_1' \, [L_3\{\text{inc } m_\lambda, \perp L_1(e_y, m_\lambda)\}, \bar{y}] = r_\lambda' \, (-\beta_\lambda^2, 0);\ Refl_z \, (r_\lambda')^2 = r_\lambda(\beta_\lambda^2, 0);$$

$$P_1' \, [L_3\{\text{inc } m_{\lambda_i}, \perp L_1(z, n_{\lambda_j})\}, \bar{y}] = s_{ij}(\beta_{\lambda_i}\beta_{\lambda_j}, 0).$$

Any element of $\Pi(\alpha_\lambda, \beta_\lambda)$ being an even expression in the β_λ with coefficients in $\Pi(\alpha_\lambda)$ is an element of $\Pi(\alpha_\lambda, \beta_\lambda^2, \beta_{\lambda_i}\beta_{\lambda_j}) = K$. We choose some point $t_{\lambda_0}(\alpha_{\lambda_0}, \beta_{\lambda_0})$ with $\beta_{\lambda_0} \neq 0$; let t_{λ_0} play the part of a third point $e_x'(0,1)$ in an affine oblique coordinate system determined by the points z, e_y, t_{λ_0}, consequently having the x-axis in common with our original orthogonal coordinate system. According to the results of Chapter III, theorem 3, the points $f(\varphi, 0)$, $\varphi \in \Pi(\alpha_\lambda, \beta_\lambda^2, \beta_{\lambda_i}\beta_{\lambda_j})$ can be constructed because we have the disposal of the operations L_1, P_1', L_2, and D^{par}.

K is the field belonging to the linear construction field upon the basis composed of the points z, e_y, t_{λ_0}, k_λ, r_λ, s_{ij}.

So far the construction of points on the x-axis.

Now let $\pi_2 \in \Pi(\alpha_\lambda,\beta_\lambda)$, π_2 being an odd expression in the β_λ with coefficients in Δ. We shall construct the point $p_2(0,\pi_2)$. We assume $\beta_1 \neq 0$. Then $d(\pi_2/\beta_1,0)$ is a constructible point on the x-axis, because $\pi_2/\beta_1 \in K$.

We assume $\pi_2 \neq \beta_1$. Then

$P_1' < L_2 \; [\text{inc } P_1' \; \{L_2(\text{inc } d, \, // \; \bar{x}), \, L_1(z,n_1)\}, \, // \; \bar{y}], \, \bar{x} > \; = p_2(0,\pi_2)$, where n_1 is the point $(1,\beta_1)$.

With the aid of the operation L_2 we easily find the point $p(\pi_1,\pi_2)$ in $R_L(B)$ from the points p_1 and p_2, if π_1, $\pi_2 \in \Pi(\alpha_\lambda,\beta_\lambda)$, π_1 being an even and π_2 being an odd rational expression in the β_λ with coefficients in $\Pi(\alpha_\lambda)$, proving the theorem.

Corollary. Given a circle in the orthogonal-linear construction field as described in theorem 9, having the equation $x^2+y^2+\tau_1 x + \tau_2 y + \tau_3 = 0$, we can construct the points $(\tau_1,0)$, $(0,\tau_2)$, $(\tau_3,0)$, because τ_1 and τ_3 are even rational expressions and τ_2 is an odd rational expression in the β_λ. Conversely we can construct the circle when these points are given. We shall not describe in detail the construction chains connected with these constructions.

Remark. Since the active operations used are univalent, every constructible object in the construction field $R_L(B)$ upon a basis B is constructible uniquely. Nevertheless, if the basis points are represented by $z(0,0)$, $e_y(1,0)$, $t_\lambda(\alpha_\lambda,\beta_\lambda)$ $\lambda \in \Lambda$, where the α_λ, β_λ are considered as parameters, the *coordinates of the points of the construction field are not always determined uniquely*; in the case of a Γ-field they are, but in the case of a P-field no longer. E.g., there are two points that may be selected to play the role of $e_x(0,1)$; in the analytical image: there are two transformations from the "absolute" coordinate system to a coordinate system with respect to which the basis points have the coordinates mentioned above. If, in a P-field, there exists an algebraic relation $\varphi'(\beta_\lambda) = \varphi''(\beta_\lambda) \neq 0$, as described in theorem 9, with respect to one of these possible coordinate systems, then it will correspond with the relation $\varphi'(\beta_\lambda) = -\varphi''(\beta_\lambda) \neq 0$ with respect to the other one. On the other hand, the place of the point (0,1) is determined uniquely, if such an algebraic relation is explicitly given.

By this an analytical element is taken into consideration; however, we shall show further on how this can be expressed in geometrical terms (Cf. section 6, pp. 51, ff).

Theorem 10. *The notions "P-field" and "Γ-field" are independent of the coordinate system, on condition that the points* (0,0) *and* (1,0) *belong to the construction field itself.*

Dem. Let be given an orthogonal-linear construction field upon a basis containing three non-collinear points $z(0,0)$, $e_y(1,0)$, $t_\lambda(\alpha_\lambda, \beta_\lambda)$, $\lambda \in \Lambda$, Λ some index system, that can be represented as a P-field or a Γ-field with respect to a fixed coordinate system. Consider a coordinate transformation (similarity transformation) to a new coordinate system: $s(\sigma_1, \sigma_2) \rightarrow z'(0,0)$, $t(\tau_1, \tau_2) \rightarrow e_y'(1,0)$; σ_1, σ_2, τ_1, $\tau_2 \in P$; σ_1 and τ_1 being even and σ_2 and τ_2 being odd rational expressions in the β_λ, in the case of a Γ-field. The transformation formulas read, when calling $\pi = (\tau_1-\sigma_1)^2 \neq (\tau_2-\sigma_2)^2$:

$$x' = \{(\tau_1-\sigma_1)\, x + (\tau_2-\sigma_2)\, y + \sigma_1^2 + \sigma_2^2 - \sigma_1\tau_1 - \sigma_2\tau_2\}.\pi^{-1},$$

$$y' = \{-(\tau_2-\sigma_2)\, x + (\tau_1-\sigma_1)\, y - \tau_1\sigma_2 + \tau_2\sigma_1\}.\,\pi^{-1}, \text{ or}$$

$$y' = \{(\tau_2-\sigma_2)x - (\tau_1-\sigma_1)\, y + \tau_1\sigma_2 - \tau_2\sigma_1\}.\,\pi^{-1}.$$

From this it follows that the new coordinates are again elements of P, and in the case of a Γ-field, that the new abscissae are even and the new ordinates are odd rational expressions of the *old* β_λ. An analogous reasoning may be held with respect to the inverse transformation. Consequently the notions P-field and Γ-field are independent of the coordinate system, provided that the points (0,0) and (1,0) belong to the construction field.

Corollary. The field P may be called the *field belonging to the orthogonal-linear construction field.* This does not mean that the construction field indeed is a P-field.

4. Adjunction operations.

The use of the *geometrical adjunction operation* A^g (Cf. p. 12) will be admitted in order to procure the basis points for an adjunctionless construction field $R_L(B)$, and in order to construct objects in an adjunctive construction field $R_{L(A)}(B)$ (Cf. p. 14).

Theorem 11. *In the case that B contains three non-collinear points the construction fields* $R_L(B)$ *and* $R_{L(A)}(B)$ *are identical; so such a basis is sufficient (with respect to L).*

Dem. If $R_L(B)$ is a P-field, the theorem is a consequence of theorem 1.

Now let us suppose that $R_L(B)$ is a Γ-field upon the basis represented by $z(0,0)$, $e_y(1,0)$, $t_\lambda(\alpha_\lambda,\beta_\lambda)$ $\lambda \in \Lambda$; at least one of the β_λ is not 0. Let $t(\tau_1,\tau_2)$ be transcendental over B; adjoin t to B forming $B' = [B,t]$; consider $R_L(B')$: this will be a new Γ'-field, because it is not possible that, by this adjunction, an algebraic relation arises, that can be written $\varphi'(\beta_\lambda, \tau_2) = \varphi''(\beta_\lambda, \tau_2) \neq 0$, where φ' is an even and φ'' is an odd rational expression in the β_λ and τ_2 together, with coefficients in $\Pi(\alpha_\lambda, \tau_1; \lambda \in \Lambda)$.

It follows from the demonstration of theorem 1 that the coordinates of a point p in $R_L(B')$ that is independent of the adjoined point t, must be elements of P. But the only points in the Γ'-field that have coordinates in P are the points of the original Γ-field. This proves the theorem.

Other kinds of bases are not sufficient. We consider the case of a basis B that consists of a set of (at least two distinct) collinear points, represented by $z(0,0)$, $e_y(1,0)$, $t_\lambda(\alpha_\lambda,0)$, $\lambda \in \Lambda$. According to the theorems derived in this chapter we find: $R_L(B) \neq R_{L(A)}(B)$. $R_{L(A)}(B)$ consists of those points $p(\pi,0)$, the coordinate π of which is an element of the field $\Pi(\alpha_\lambda; \lambda \in \Lambda)$ and of the line connecting them. In this case we shall call $R_{L(A)}(B)$ the *orthogonal-linear construction field upon B*; it can be described as the Γ-field upon B (all β_λ are 0).

Remark. In section 7 the (geometrical) *determinate adjunction operation A^d* will be used in a special case (Cf. Ch. I, p. 12).

5. Construction geometry of normals. (*i*)

In reflection geometry we consider the construction geometry (R,N) built with the abstract means of construction suggested by the use of the practical constructive instruments *ruler* and *right square* (German: rechter Zeichenwinkel.). The latter admits of two kinds of use; in the present chapter only the first kind is significant: *given a point a and a line $\bar{a}$ the application of the right square produces the line passing through a and normal to $\bar{a}$.*

The adjunctionless operation set N will consist of the following construction postulates:

Active operations. (Univalent operations.)

L_1 (Chapter II, p. 22); P_1' (Chapter III, p. 28);

L_3: *There exists an operation* L_3, *which when applied to a point* a *and a line* $\bar{a}$, *produces the line* L_3 (*inc* a, $\perp$ $\bar{a}$).

Decision operations: D^{pp}, D^{ll} (Chapter II, p. 22);

Selection operation: S_1 (Chapter II. p. 22).

N may be extended to $N(A)$ by adding the geometrical adjunction operations A^g and A^d (Cf. Ch. I., p. 12). The adjunction operations may only serve for the well known purposes, indicated before.

The active operation L_3 *is non-conditional.*
The only *objects* of R considered are points and lines.

As a consequence of the possibility of the active operation L_3 and the decision operation D^{ll} we have the decision operation theorem
D^{norm}: *It is possible to decide whether two distinct lines are normal or not.*

Furthermore the operation theorems hold:

D^{pl}: *It is possible to decide whether a point* a *and a line* $\bar{a}$ *are incident or excident.*

Dem. As in Chapter II, p. 23.

L_2: *It is possible to construct the line passing through a given point* a, *parallel to a given line* $\bar{a}$ (a *exc* $\bar{a}$).

Dem. L_3 {inc a, $\perp$ L_3 (inc a, $\perp$ $\bar{a}$)} $= \bar{b}$, this being the parallel required.

D^{par}: *It is possible to decide whether two lines are parallel or not.*

Dem. This is a consequence of the possibility of the active operation L_2 and the decision operations D^{ll} and D^{pl}.

Refl$_{\bar{b}}$(*a*): *It is possible to construct the reflection point of a given point* a *with respect to a given line* $\bar{b}$.

Dem. If a inc $\bar{b}$, a is the point required If a exc $\bar{b}$, we proceed:
P_1' {$\bar{b}$, L_3 (inc a, $\perp$ $\bar{b}$)} $= b_2$; let b_1 be some point of $\bar{b}$, $b_1 \neq b_2$.

$$P_1' \{L_3(\text{inc } b_1, \perp \bar{b}), L_2 (\text{inc } a, // \bar{b})\} = c;$$

$$P_1' [L_2 \{\text{inc } b_1, // L_1(c,b_2)\}, L_3(\text{inc } a, \perp \bar{b})] = \text{Refl}_{\bar{b}}(a).$$

Refl$_b$(*a*): *It is possible to construct the reflection point of a given point* a *with respect to a given point* b.

Dem. If $a = b$, $\text{Refl}_b(a) = a$; if $a \neq b$, the reflection point required is obtained by operating $\text{Refl}_{\bar{b}}(a)$, where $\bar{b} = L_3\{\text{inc } b, \perp L_1(a,b)\}$.

We find: all operations of L (except C_1) are possible in the construction geometry (R,N). Conversely all operations of N are possible in (R,L).

Finally, because in the "sufficiency"-part of the demonstration of theorem 9 the operation C_1 did not play a role, we have:

Theorem 12. *When* $z(0,0)$, $e_y(1,0)$, $t_\lambda(\alpha_\lambda,\beta_\lambda)$, $\lambda \in \Lambda$, *constitute a basis* B *containing three non-collinear points* (*sufficient basis*) *the construction field* $R_N(B)$ *is the orthogonal-linear construction field* (*without the circles*) *upon* B, *as described in section* 4.

An algebraical proof of this theorem can be given in such a manner that it can be compared with the demonstration of theorems 8 and 9. However, we have preferred to show that the operation sets L and N are equivalent (with the exception of the construction of circles).

It was TIETZE [2] who mentioned already a theorem corresponding with our theorem 12.

In the case of a basis B consisting of a set of (at least two distinct) collinear points, $R_N(B)$ contains these basis points, the line connecting them, and a set of mutually parallel normals to this line through the basis points; so $R_N(B)$ has an entirely different structure than $R_L(B)$.

It is clear that in this case $R_N(B) \neq R_{N(A)}(B) \neq R_{L(A)}(B)$.

6. Orientation in the orthogonal-linear construction field.

Definitions. A non-collinear point triplet a,b,c is called a *triangle.* (The letters a, b, c may be written down independently of the order of succession.)

An ordered non-collinear point triplet a,b,c is called an *oriented triangle.* Such an ordered triangle will be denoted with the aid of an arrow over the letters, defining: $\overrightarrow{a,b,c} = \overrightarrow{b,c,a} = \overrightarrow{c,a,b} = \overleftarrow{a,c,b} = \overleftarrow{c,b,a} = \overleftarrow{b,a,c}$.

An *isosceles right-angled triangle* (shortly i.r.t.) is a triangle a_1,a_2,a_3 with the properties: $L_1(a_i, a_j)$ is normal to $L_1(a_j, a_k)$ while $a_i, a_j \simeq a_k, a_j$ for at least one set of values $i, j, k = 1, 2, 3$; $i \neq j \neq k \neq i$.

Definition. Let B be a given basis in R; the construction field $R_L(B)$ is called *orientable, if and only if it contains an isosceles right-angled triangle.*

Theorem 13. *Let* $z(0,0)$, $e_y(1,0)$, $t_\lambda(\alpha_\lambda,\beta_\lambda)$, $\lambda \in \Lambda$, *represent a basis* B *in* R; *then the orthogonal-linear construction field* $R_L(B)$ *is orientable if and only if a point* $e_x(0,1)$ *is constructible from* B *with the aid of the operations of* L.

Dem. The sufficiency is evident: e_y, z, e_x is an i.r.t. belonging to $R_L(B)$.

Necessity.

Let R_L contain an i.r.t.: a_1,a_2,a_3; apply D^{norm} to $L_1(a_i,a_j)$ and $L_1(a_k,a_j)$, $i \neq j \neq k = 1, 2, 3$. Let $L_1(a_1,a_3)$ be normal to $L_1(a_2,a_3)$; then a_3 will be called c; describe the circle $C_1(c,a_1)$; let a figure T consist of the point c and the circle; then $a_1 \,\&\, a_2 = a \,\&\, b$ are T-equivalent; now we select: $S_1(a \,\&\, b) = d_1$ *and* d_2. $\longrightarrow$

By this selection the i.r.t. has been oriented: d_1, c_1, d_2.

This selection once made, the points d_1 and d_2 will play an exactly determined part in the following construction chain. Up from here this construction chain for a point $e_x(0,1)$ will have a *unique result.*

We call $L_1(z,e_y) = \bar{y}$, and $L_3(\text{inc } z, \perp \bar{y}) = \bar{x}$;

if $c = z$ and $d_1 = e_y$, then $d_2 = e_x(0,1)$;

if $c = z$ and d_1 inc $L_1(z,e_y)$, $d_1 \neq e_y$, then

$$P_1' \, [L_2\{\text{inc } e_y, // L_1(d_1,d_2)\}, \bar{x}] = e_x(0,1);$$

if $c = z$ and d_1 exc $\bar{y}$,

$$L_1 \, [P_1' \, \{L_2(\text{inc } d_1, // \, \bar{x}), \bar{y}\}, P_1' \, \{L_2(\text{inc } d_2, // \bar{y}), \bar{x}\}] = \bar{s};$$

$$P_1' \, \{L_2(\text{inc } e_y, // \, \bar{s}), \bar{x}\} = e_x(0,1);$$

if $c \neq z$, the construction is reduced to one of the cases mentioned above by a *parallel translation* of the triangle d_1,c,d_2: (we use the notation S_2 mentioned in Chapter I, p. 11).

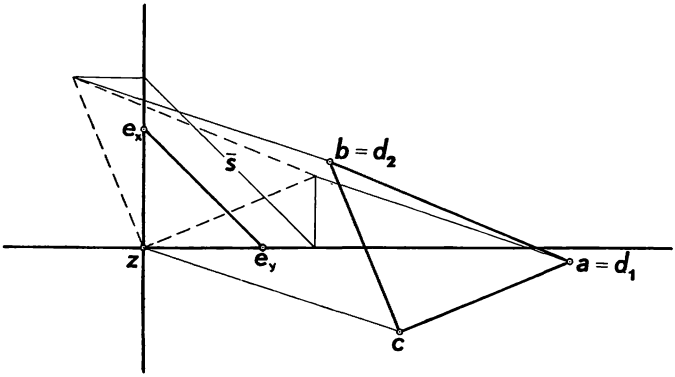

FIG. 6

$$S_2\,(d_1 \;\&\; d_2) = d_i \text{ and } d_j \text{ so that } d_i \text{ exc } L_1(z,c);$$
$$P_1'\,[L_2\{\text{inc } z, // L_1(c,d_i)\}, L_2(\text{inc } d_i, // L_1(z,c)\}] = d_i';$$
$$P_1'\,[L_2\{\text{inc } z, // L_1(c,d_j)\}, L_2(\text{inc } d_i', // L_1(d_i,d_j)\}] = d_j';$$

now we are back in one of the three cases mentioned above and we complete the construction of $e_x(0,1)$ according to the above construction chains by substituting d_1' and d_2' for d_1 and d_2 respectively, proving the theorem.

Corollary 1. The orthogonal-linear construction field $R_L(B)$ upon a basis B consisting of the points $z(0,0)$, $e_y(1,0)$, $t_\lambda(\alpha_\lambda,\beta_\lambda)$, $\lambda \in \Lambda$, is a P-field if and only if $R_L(B)$ is orientable.

Corollary 2. $R_L(B)$ is a P-field if and only if it contains an i.r.t.

Corollary 3. Although the decision operations of L allow us to decide whether a triangle is an isosceles right-angled one or not, such a triangle is not constructible in the construction geometry (R,L) in general; in a Γ-field it cannot be constructed.

Corollary 4. In the demonstration of theorem 13 we have shown that in the construction geometry (R,L) the *parallel translation* is performable.

Corollary 5. The permutations of the T-equivalent points $a_1 \;\&\; a_2$ give rise to two possible results of the construction chain for a point (0,1). These resulting points are distinct because the characteristic p of our geometry is not equal to 2.

Definition. *Two oriented isosceles right-angled triangles* $\overrightarrow{d_1,c,d_2}$ *and* $\overrightarrow{d_1'',c'',d_2''}$ *have similar orientation if and only if the point* e_x *obtained (according to the construction chain of theorem* 13) *from* $\overrightarrow{d_1,c,d_2}$ *coincides with the point* e_x'' *obtained from* $\overrightarrow{d_1'',c'',d_2''}$

The property of having similar orientation is *reflexive, symmetric,* and *transitive.* So we may speak about the *orientation* of an oriented i.r.t. with respect to another oriented i.r.t.

Corollary. If e_x is the point constructed from an oriented i.r.t. $\overrightarrow{d_1,c,d_2}$, according to the construction chain of theorem 13, the oriented triangles $\overrightarrow{e_y,z,e_x}$ and $\overrightarrow{d_1,c,d_2}$ have similar orientation.

Definition. Two oriented isosceles right-angled triangles are said to have *opposite orientation,* if they do not have similar orientation.

Corollary. The oriented isosceles right-angled triangles $\overrightarrow{d_1,c,d_2}$ and $\overrightarrow{d_2,c,d_1}$ have opposite orientation.

Definition. *An orthogonal-linear P-field is called oriented if and only if it contains an oriented isosceles right-angled triangle.*

From all this we may draw the following conclusions.

Let be given a basis B consisting of the points $z(0,0)$, $e_y(1,0)$, $t_\lambda(\alpha_\lambda,\beta_\lambda)$, $\lambda\in\Lambda$, and containing an oriented i.r.t. $\overrightarrow{a_1,a_2,a_3}$, $a_1\hat{a}_2a_3 = 90^o$; then the point e_x such that e_y,z,e_x is an oriented i.r.t. while $\overrightarrow{e_y,z,e_x}$ and $\overrightarrow{a_1,a_2,a_3}$ have similar orientation, and the point d_x such that e_y,z,d_x is an oriented i.r.t. while $\overrightarrow{e_y,z,d_x}$ and $\overrightarrow{a_1,a_2,a_3}$ have opposite orientation, are constructible uniquely and, so to say, separately.

An exact construction problem may run: to construct *the* point $e_x(0,1)$ so that $\overrightarrow{e_y,z,e_x}$ and $\overrightarrow{a_1,a_2,a_3}$ have similar orientation. Then this problem is solvable uniquely. In connection with the remark of section 3, p. 47, we find that now there must exist an algebraic relation $\varphi'(\beta_\lambda) = \varphi''(\beta_\lambda) \neq 0$, where φ' is an even and φ'' is an odd rational expression in the β_λ, with coefficients in $\Pi(\alpha_\lambda;\ \lambda\in\Lambda)$. Indeed, that relation is found when using the transformation formulas mentioned on p. 48: we will represent the coordinates of the a_i by $a_1(\alpha_1',\beta_1')$, $a_2(\alpha_2',\beta_2')$, $a_3(\alpha_3',\beta_3')$; then we find:

$$(\alpha_3'-\alpha_2')^2+(\beta_3'-\beta_2')^2=-(\beta_3'-\beta_2')\,\alpha_1'+(\alpha_3'-\alpha_2')\,\beta_1'-\alpha_3'\beta_2'+\beta_3'\alpha_2\neq 0.$$

Conversely, if B is a basis consisting of the points $z(0,0)$, $e_y(1,0)$, $t_\lambda(\alpha_\lambda,\beta_\lambda)$, $\lambda\in\Lambda$, and the point $e_x(0,1)$ is assumed to be constructible uniquely, then each i.r.t. a_1,a_2,a_3 ($a_1\hat{a}_2a_3 = 90^o$) in the construction field $R_L(B)$ (that is assumed a P-field) can be oriented uniquely in such a manner that the triangles $\overrightarrow{a_i,a_2,a_j}$ and $\overrightarrow{e_y,z,e_x}$ have similar orientation. So we find:

Theorem 14. *Let $z(0,0)$, $e_y(1,0)$, $t_\lambda(\alpha_\lambda,\beta_\lambda)$, $\lambda\in\Lambda$ constitute a basis of an orthogonal-linear construction field that is a P-field. If $R_L(B)$ is oriented, the points $e_x(0,1)$ and $d_x(0,-1)$ can be described in such a manner that they can be constructed uniquely and separately; con-*

versely, if the points $e_x(0,1)$ *and* $d_x(0,-1)$ *can be constructed uniquely, both possible orientations (with respect to the oriented triangle* $\overrightarrow{e_y,z,e_x}$*) of any isosceles right-angled triangle can be determined and, so to say, "constructed" uniquely.*

Less correct, but shorter, we say: the points e_x and d_x are constructible uniquely and separately in a P-field if and only if the P-field is oriented.

As a consequence of the definitions of this section and of theorem 13 we find the *decision operation theorem*:

D^{ort}: *It is possible to decide whether two oriented isosceles right-angled triangles have similar or opposite orientation.*

Remark 1. If the orthogonal-linear construction field is a Γ-field, each point of it can be represented uniquely by its coordinates; but if it is a P-field the coordinates do not determine the points uniquely. The place of the point $e_x(0,1)$ is not determined (uniquely) with respect to the coordinates of the basis points. After the introduction of the concepts of similar and opposite orientation for oriented isosceles right-angled triangles, we are able to describe the objects of a P-field uniquely. On the other hand, the objects of a P-field are constructible uniquely, because the operations of L are univalent. With regard to exact construction problems worded with respect to the basis of orthogonal-linear construction field we find: any object that can be described uniquely by geometrical or analytical terms, can be constructed uniquely with the aid of the operations of L, and conversely.

Remark 2. In the orthogonal-linear construction field upon a sufficient basis a set of two mutually non-parallel right angles can be constructed. Such a set determines *the pair of isotropic points* i_1 & i_2. (These points do not belong to the construction field in general.)

The question whether the orthogonal-linear construction field upon a basis B is a Γ-field or a P-field depends essentially on the structure of B.

If the basis is such that the orthogonal-linear construction field $R_L(B)$ is a Γ-field, the objects of $R_L(B)$ are invariant with respect to the permutations of i_1 & i_2. Already TIETZE [2] has shown that the Γ-field is exactly the subfield of the P-field, that is invariant with respect to the permutations of i_1 & i_2. As a consequence of this statement we have: *The orthogonal-linear construction field upon a sufficient basis B in the construction geometry (R,L) is a P-field if and only if B has such properties that the isotropic points can be distinguished.*

Remark 3. It is a consequence of theorems 9 and 10 that similar and opposite orientation can be defined and decided (in an oriented P-field) for triangles that can be represented (after a suitable similarity transformation) by the points with coordinates (1,0), $(\alpha,0)$, $(0,\beta)$, whereby the remaining basis points are named $t_\lambda(\alpha_\lambda,\beta_\lambda)$, $\lambda \in \Lambda$, while there is a *known* algebraic relation $\varphi'(\beta,\beta_\lambda) = \varphi''(\beta,\beta_\lambda) \neq 0$, φ' being an even and φ'' being an odd rational expression in the β,β_λ together, with coefficients in $\Pi(\alpha,\alpha_\lambda;\ \lambda \in \Lambda)$, β occurring at least in one of the functions φ' or φ'' In that case an isosceles right-angled triangle can be constructed uniquely from such a triangle and the remaining data.

Similar and opposite orientation can be defined and decided (both in P-field and in Γ-field) for such two similar (and congruent) triangles that there is a transformation from the one to the other, *constructible* in (R,L).

We will not elaborate all details.

Remark 4. When B is a basis with such a structure that $R_L(B)$ is a non-oriented P-field, B may generally contain two points a, b and point *couple* c_i, $i = 1, 2$, that can be represented (after a suitable similarity transformation) by $a(1,0)$, $b(\alpha,0)$, $c_i(0,\pm\beta)$, whereby the remaining basis points are named $t_\lambda(\alpha_\lambda, \pm\beta_\lambda)$, $\lambda \in \Lambda$, while there is a known algebraic relation $\varphi'(\beta,\beta_\lambda) = \pm\varphi''(\beta,\beta_\lambda) \neq 0$, φ' being an even and φ'' being an odd rational expression in the β,β_λ together, with coefficients in $\Pi(\alpha,\alpha_\lambda;\ \lambda \in \Lambda)$. c_1,c_2 is considered given as a point *couple*, and not as a couple of points. Because we consider only bases that consist of *points*, and because only *points* will be adjoined by adjunction operations, we shall never come across such bases. If a triangle a,b,c is given (a, b, c considered as points), it is automatically oriented by the order of succession in which the letters are written down. The only interesting question is whether we can decide concerning similar and opposite orientation of two triangles, or not.

So it will no longer be necessary to write $\overrightarrow{a,b,c}$ for a,b,c.

As a consequence of this remark we find that it is not necessary to introduce construction operations such as: an operation that produces two points separately from a given point couple, an operation that produces a point couple from two points given separately, an operation that produces an oriented triangle from a non-oriented triangle.

8. Automorphisms of orthogonal-linear construction fields.

Definition. *An automorphism of a construction field, say $R_L(B)$, is a one-to-one mapping of $R_L(B)$ onto itself, points onto points, lines onto lines, circles onto circles, etc., all relations between the objects being invariant.*

Let B be a basis composed of the points $z(0,0)$, $e_y(1,0)$, $t_\lambda(\alpha_\lambda,\beta_\lambda)$, $\lambda \in \Lambda$, in such a manner that $R_L(B)$ is a Γ-field. Next we suppose that a point $e_x(0,1)$ can be constructed with the aid of certain operations outside L. Then we adjoin e_x to B, forming B' by applying the *determinate* (geometrical)adjunction operation A^d.

The orthogonal-linear construction field $R_L(B')$ is the P-field that is called the P-field *including* the Γ-field $R_L(B)$. So $p(\pi_1,\pi_2) \in R_L(B')$ if and only if $\pi_1,\pi_2 \in \Pi(\alpha_\lambda,\beta_\lambda)$.

Because $R_L(B)$ is a Γ-field there does not exist an algebraic relation $\varphi' = \varphi'' \neq 0$, where φ' is an even and φ'' is an odd rational expression in the β_λ with coefficients in $\Pi(\alpha_\lambda)$. (Cf. Theorem 9). This implies that any element ε of $\Pi(\alpha_\lambda,\beta_\lambda)$ can be uniquely written: $\varepsilon = \varepsilon' + \varepsilon''$, where ε' is an even and ε'' is an odd rational expression in the β_λ with coefficients in $\Pi(\alpha_\lambda)$, because $F(\beta_\lambda) = \frac{1}{2}\{F(\beta_\lambda) + F(-\beta_\lambda)\} + \frac{1}{2}\{F(\beta_\lambda) - F(-\beta_\lambda)\}$.

Theorem 15. *Let p, $\bar{r}$, $\bar{s}$, $\bar{\bar{t}}$ represent the objects of $R_L(B')$. The mapping $p(\pi_1'+\pi_1'', \pi_2'+\pi_2'') \rightarrow p_0(\pi_1'-\pi_1'', -\pi_2'+\pi_2'')$, $\bar{r}(y = \varrho_1'+\varrho_1'')\, x+\varrho_2'+\varrho_2'') \rightarrow \bar{r}_0(y = (-\varrho_1'+\varrho_1'')\, x - \varrho_2'+\varrho_2'')$, $\bar{s}\,(x = \sigma'+\sigma'') \rightarrow \bar{s}_0(x = \sigma'-\sigma'')$, $\bar{\bar{t}}(x^2+y^2+(\tau_1'+\tau_1'')\, x + (\tau_2'+\tau_2''))\, y + \tau_3'+\tau_3''=0) \rightarrow \bar{\bar{t}}_0(x^2+y^2+(\tau_1'-\tau_1'')\, x + (-\tau_2'+\tau_2'')y + \tau_3'-\tau_3'' =0)$, is an automorphism of $R_L(B')$. An object of $R_L(B')$ is invariant with respect to this automorphism if and only if it belongs to the Γ-field $R_L(B)$.*

Dem. The first statement follows from computation; all relations of incidence, parallelism, normality, congruence remain invariant. The proof follows from the fact that, if we find an algebraic relation $\varphi'(\beta_\lambda) = \varphi''(\beta_\lambda)$, where φ' is an even and φ'' is an odd rational expression in β_λ with coefficients in $\Pi(\alpha_\lambda)$, necessarily $\varphi'(\beta_\lambda) = 0$ and $\varphi''(\beta_\lambda) = 0$.

The second statement is an immediate consequence of the definition of Γ-field.

This automorphism and the identical automorphism together constitute a two-cyclic group.

We say that the automorphisms *leave* $R_L(B)$ *objectwise fixed.*

If $B_0 \subset B$, the orthogonal-linear construction field $R_L(B_0)$ is also left fixed objectwise by the automorphisms of $R_L(B')$.

9. Ruler constructions in reflection geometry when a set of two non-parallel right angles is given.

It is a well known fact that given the figure of two mutually non-parallel right angles (couples of normal lines) a large number of orthogonal-linear constructions can be performed by using the ruler only. We should say: in the construction geometry of normals (R,N) the operation L_3 need be used only *twice*. Nevertheless, it will not be sufficient to put this limiting restriction into (R,N). We shall need the operation L_2 (producing parallels) as a *postulated operation* in order to reach the result that the possible constructions will lead to *orthogonal-linear construction fields*, because the possibility of L_2 can not be reduced to L_3 any more.

The operation set L'' of postulated operations will be introduced: L_1, P_1', L_2, D^{pp}, D^{ll}, S_1; only those bases will be considered that contain four points a,b,c,d so that, e.g., $a\hat{b}d = a\hat{c}d = 90°$ Without proof we mention that all bases considered are sufficient.

The possibility of constructing normals depends on the well known perpendicular theorem in triangles; we shall not deal with all details.

Theorem 16. *When* $z(0,0)$, $e_y(1,0)$, t_λ $(\alpha_\lambda,\beta_\lambda)$, $\lambda \in \Lambda$, *constitute a basis* B *containing four points as described above, the construction field* $R_{L''}(B)$ *is the orthogonal-linear construction field upon* B.

If the basis B of the construction field in R contains a *square*, then $R_L''(B)$ is also the orthogonal-linear construction field. (Because a square is a parallelogram too, we may replace the active operation postulate L_2 in the set L'' by the decision operation postulate D^{par}, Cf. theorem 5, p. 35).

Let B contain four points a_1, a_2, a_3, a_4 which constitute a square.

To build the construction field we continue:

$S_1(a_1 \,\&\, a_2 \,\&\, a_3 \,\&\, a_4) = z(0,0)$; the remaining points are called $b_1 \,\&\, b_2 \,\&\, b_3$; we shall use the notation S_2:

$$S_2\ (b_1 \,\&\, b_2 \,\&\, b_3) = b_1 \textit{ so that } L_1(z,b_1) \,//\, L_1(b_2, b_3);$$

$$S_2\ (b_2 \,\&\, b_3) = b_2 \textit{ so that } L_1(z,b_2) \,//\, L_1(b_1,b_3);$$

$P_1{}' \,\{L_1(z,b_3),\ L_1(b_1,b_2)\} = s$, the point of intersection of the diagonals.

We obtain a set of two mutually non-parallel right angles.

The triangle b_1, z,b_2 is isosceles right-angled. b_1 and b_2 are T-equivalent with respect to the set T consisting of b_1 and z.

Next: $S_1(b_1 \,\&\, b_2) = e_y(1,0)$; finally it is clear that the construction field will be the orthogonal-linear construction field, in particular a P-field.

It is a well known fact, that given a *square* more constructions are possible when using the ruler only (and D^{par}!), than in the case that a couple of mutually non-parallel right angles is given. What is the difference? The answer to this question is simply this: in the latter case the construction field $R_{L''}(B)$ may be either a P-field or a Γ-field, and in the former case $R_{L''}(B)$ is always a P-field.

CHAPTER V

CONSTRUCTIONS IN CONGRUENCE GEOMETRY USING RESTRICTED BIVALENT OPERATIONS

1. Objects, relations, operations.

A geometrical system satisfying the axioms on *incidence, parallelism,* and a set of such axioms on *congruence,* that they are significant in an analytical geometry over any field $\varDelta$ with the property: if $\vartheta \in \varDelta$, then also $\sqrt{1+\vartheta^2} \in \varDelta$, is called a *congruence geometry C.*

Again we shall not set up such a sufficient set of congruence axioms; the introduction of the required notions may happen in the same manner as in Chapter IV, section 1, with the only modification that the field $\varDelta$ considered must have the property mentioned above. We assume it well known that consequently $\varDelta$ must have characteristic $p = 0$.

The *totally real* field Ω, the elements of which are derived from those of the field $\varPi$ of the rational numbers by a finite number of additions, subtractions, multiplications, divisions, and of operations $\vartheta \to \sqrt{1+\vartheta^2}$, ϑ being an element already constructed, is the smallest field contained in the field underlying a congruence geometry.

Congruence geometry is the HILBERT geometry without order, where HILBERT geometry is the geometry based on the first four axiom groups of the HILBERT axiom system, mentioned in the "Grundlagen der Geometrie"

Objects: points, lines, circles; relations: coincidence, incidence, parallelism, congruence, normality.

We shall say that an object is *described* with regard to a figure F by summing up all relations existing between this object and the objects of F. In congruence geometry C it will not always be possible to describe an object of C with regard to a certain figure F *uniquely.*

In this chapter we will use the notation $\sqrt{1+\vartheta^2}$ for "*a* root of the equation $x^2 = 1 + \vartheta^2$"; e.g., given two points called z(0,0) and $e_y(1,0)$, we shall not be able to speak about *the* point $(\sqrt{2},0)$, but only about *a* point $(\sqrt{2},0)$.

We shall introduce the adjunctionless operation set Q, that will particularly contain the *branching operation* producing both points of intersection of a circle and a line *incident with the centre of the circle*. Q will consist of the following construction postulates.

Active operations.

L_1 (Chapter II, p. 22), P_1' (Chapter III, p. 28), C_1 (Chapter IV, p. 41); L_3 (Chapter IV, p. 50); (Univalent operations).

P_4': *There exists an operation* P_4', *which when applied to a circle* $\bar{\bar{a}}$ *with centre* a *and to a line* $\bar{a}$, a *inc* $\bar{a}$, *produces both points of intersection of* $\bar{\bar{a}}$ *and* $\bar{a}$ *simultaneously*: $P_4'(\bar{a},\bar{\bar{a}}) = b_1 \mathbin{\&} b_2$.
(Bivalent operation).

Decision operations: D^{pp}, D^{ll} (Chapter II, p. 22).

Selection operation: S_1 (Chapter II, p. 22).

Remark. The notation $P_4'(\bar{a},\bar{\bar{a}}) = b_1 \mathbin{\&} b_2$ does not mean that the points $b_1 \mathbin{\&} b_2$ are produced as a *couple*; we will consider them as two *points*. The names b_1 and b_2 are given to them arbitrarily; if it is convenient to interchange those names afterwards, e.g., after the application of certain decision operations, this may always happen.

Example: $P_4'\,\{L_1(b,b),\ C_1(\dot{b},a)\} = c_1 \mathbin{\&} c_2$; one of these points coincides with a; apply D^{pp} to c_1 and a, and to c_2 and a; then we will write: $S_2(c_1 \mathbin{\&} c_2) = c_1$ *and* c_2 *so that* $c_2 = a$.

2. Construction theorems.

D^{pl}: The decision whether a point and a line are incident or excident is possible in the same manner as in the preceding construction geometries.

D^{pc}: We decide whether a point a is incident or excident with a circle $C_1(\dot{b},\ c)$ applying D^{pp} to a and $S_1[P_4'\{L_1(a,b),\ C_1(\dot{b},c)\}] = p_1$ *and* p_2, if $a \neq b$.

D^{cc}: The decision whether two circles coincide or not, runs as in Chapter IV, p. 42.

L_2: Given a point a and a line $\bar{b}$, a exc $\bar{b}$, then we construct the line through a *parallel* to $\bar{b}$:

$L_3\,\{\text{inc } a \perp L_3\,(\text{inc } a,\ \perp \bar{b})\}$ is the parallel required.

D^{par}: As a consequence of the possibility of the operations L_2 and D^{ll} we can decide whether two distinct lines are parallel or not.

D^{norm}: Since the operations L_3 and D^{ll} are available, we can decide whether two distinct lines are normal or not.

$Refl_{b}(a)$: Given two distinct points a and b, then $S_2\ [P_4'\{L_1(a,b),\ C_1\ (\dot{b},a)\}] = c$ *so that* $c \neq a$; then c is the reflection point of a with respect to b.

$Refl_{\bar{b}}(a)$: Given a point a and a line $\bar{b}$, a exc $\bar{b}$; then $P_1'\ \{L_3(\text{inc}\ a,\ \perp\ \bar{b}),\ \bar{b}\} = b$; $Refl_{b}(a) = c$; then c is the reflection point of a with respect to b.

Conclusion: we find that all operations of the operation set L (Of Chapter IV; "orthogonal-linear" operations) can be performed with the aid of the postulated operations of Q. We may say that the reflection construction geometry (C,L) is imbedded in congruence construction geometry (C,Q).

Consequently the remaining construction theorems of Chapter IV hold also in the present construction geometry: P_2, P_3, C_2, D^{ort}.

Moreover, the following remark must be made concerning the introduction of L_3 as a postulate. There is a construction for L_2 without the use of L_3, and next there is a construction for L_3 with the aid of the remaining operations of Q, *if three non-collinear points are given*. We prefer the introduction of L_3 as a postulate, so that, given two distinct points, a third point outside the line connecting them can be *constructed* with the aid of the operations of Q; consequently we have: given a basis B of a construction field, B consisting of (at least two distinct)collinear points, the points of the orthogonal-linear construction field $C_{L(A)}(E)$ can be constructed in the construction geometry (C,Q), without the use of an adjunction operation. (Cf. Chapter IV, p. 49).

Finally a new operation will be derived:

P_5': *Given two non-collinear points a, b, it is possible to construct the points of intersection of the circles* $C_1(\dot{a},b)$ *and* $C_1(\dot{b},a)$ (*the one incident with the other's centre*). (*Bivalent operation*).

Dem. We shall use the operation M, producing the midpoint of a given couple of points; it has been described in Chapter III, p. 33; in the present geometry M is possible since the characteristic p is equal to 0.

$S_1 < P_4'\ [L_3\,\{\text{inc}\ a,\ \perp L_1(a,b)\},\ C_1(\dot{a},b)] > \, = d$;

$S_1 < P_4'\ [L_3\,\{\text{inc}\ d,\ \perp L_1(b,d)\},\ C_2(\dot{d},\widehat{ab})] > \, = e$;

$M\ (b,e) = f$; $M\ (a,b) = m$; $P_4'\ [L_3\ \{\text{inc}\ m,\ \perp L_1(a,b)\},\ C_2(\dot{m},\widehat{bf})] =$ s_1 & s_2, being the points required.

3. Construction fields.

Consider the (adjunctionless) construction field $C_Q(B)$ in the geometry C under the operations of Q upon a basis B of given objects (in particular: points) of C.

Remark. In this chapter the orthogonal-linear construction field will always be denoted by $C_L(B)$, although in the case of a basis consisting of collinear points the notation $C_{L(A)}(B)$ should be correct. When speaking about the operation set L of Chapter IV, it will be assumed that L_3 as a postulate has been added for the case of bases consisting of collinear points.

Consider a basis containing two distinct points; the basis points may be represented by $z(0,0)$, $e_y(1,0)$, $t_\lambda(\alpha_\lambda,\beta_\lambda)$, $\lambda \in \Lambda$. (The coordinate system is an orthogonal cartesian one.)

Let $\Omega(\alpha_\lambda,\beta_\lambda)$ be the field, the elements of which are derived from those of the field $\Pi(\alpha_\lambda,\beta_\lambda;\ \lambda \in \Lambda)$ by a finite number of additions, subtractions, multiplications, divisions, and operations $\vartheta \rightarrow \sqrt{1+\vartheta^2}$.

Then, if B is a basis as described just now, we have:

Theorem 17. *A point $p(\pi_1,\pi_2)$ belongs to the construction field $C_Q(B)$ if and only if its coordinates π_1, π_2 are elements of the field $\Omega(\alpha_\lambda,\beta_\lambda)$.*

Dem. *Necessity.*

The basis points satisfy the condition of the theorem. Let $s(\sigma_1,\sigma_2)$, $t(\tau_1,\tau_2)$, $c(\gamma_1,\gamma_2)$ be three points with coordinates in $\Omega(\alpha_\lambda,\beta_\lambda)$; then the coordinates of the points of intersection of the line $L_1(s,c)$ with the equation $(\sigma_1-\gamma_1)\ (y-\sigma_2) = (\sigma_2-\gamma_2)\ (x-\sigma_1)$, and the circle $C_1(c,t)$ with the equation $(x-\gamma_1)^2+ (y-\gamma_2)^2 = (\tau_1-\gamma_1)^2+(\tau_2-\gamma_2)^2$ similarly are elements of $\Omega(\alpha_\lambda,\beta_\lambda)$.

The remaining construction operations are reflected by rational operations with respect to the coordinates and the equation coefficients.

Sufficiency.

The place of *the* point (0,1) is not determined (uniquely); there are two points that may play the part of (0,1); in analytical language: there are two transformations from the "absolute" coordinate system to a coordinate system with respect to which the given basis points have the coordinates mentioned in the theorem. We construct *a* point $e_x(0,1)$:

$$S_1 < P_4' \, [L_3 \{\text{inc } z, \perp L_1(z, e_y)\}, C_1(\dot{z}, e_y)] > = e_x(0,1).$$

Henceforth the rational operations are constructible uniquely in *the linear construction field* upon the basis $B' = [B, e_x]$ (The *adjunction* of e_x to B is a determinate one, when we consider the operation set L; with respect to Q no adjunction need take place.)

Next, let a point $a(\vartheta,0)$ of $C_Q(B)$ be given. For certain particular structure of B it could be possible that a point $b(\sqrt{1+\vartheta^2},0)$ could be constructed by the use of the univalent operations of the set L. We shall not go into this question; in the general case such a situation will not occur. So in the following we assume that b cannot be constructed uniquely by the use of the operations of L only; then: $L_1(z,e_y) = \bar{y}$; $P_1'\{L_3(\text{inc } a, \perp \bar{y}), L_2(\text{inc } e_x, // \, \bar{y})\} = c$;
$P_4' \{\bar{y}, C_1(\dot{z},c)\} = b_1 \,\&\, b_2$.

$\sqrt{1+\vartheta^2}$ means: *a* root of the equation $x^2 = 1 + \vartheta^2$; each of the points $b_1 \,\&\, b_2$ may be selected to be $(\sqrt{1+\vartheta^2},0)$; then the other one is $(-\sqrt{1+\vartheta^2},0)$: $S_1(b_1 \& b_2) = b_1(\sqrt{1+\vartheta^2},0)$ *and* $b_2(-\sqrt{1+\vartheta^2},0)$.

Each of the points is *a* solution b required; both points together are *the* solutions. Combining the rational operations in the linear construction field and the construction for the operation $\vartheta \to \sqrt{1+\vartheta^2}$ derived above, we may conclude that, for any two elements π_1, π_2 of $\Omega(\alpha_\lambda, \beta_\lambda)$, a point p and even all points p can be constructed for which the coordinates may be chosen as (π_1, π_2); the construction of a line and a circle, with given equation coefficients (given in the form of point coordinates) runs in the case of a line by linear operations (affine plane), and in the case of a circle by orthogonal-linear operations (reflection geometry); we find that all algebraic operations possible in $\Omega(\alpha_\lambda, \beta_\lambda)$ (concerning point coordinates and equation coefficients) are constructible in (C, Q), proving the theorem.

Remark. Without proof we state:

$\Omega(\alpha_\lambda, \beta_\lambda)$ is independent under coordinate transformation (similarity transformation) in $C_Q(B)$, i.e., on condition that the new points (0,0) and (1,0) belong to $C_Q(B)$; it is called *the field belonging to* $C_Q(B)$.

4. Remarks and examples.

We point out that many points of $C_Q(B)$ cannot be described uniquely with regard to the basis B. When we say that π_1 and π_2 are elements of $\Omega(\alpha_\lambda, \beta_\lambda)$ this means (in this non-ordered geometry) that π_1 and π_2 are roots of certain irreducible polynomials with coefficients in $\Pi(\alpha_\lambda,\beta_\lambda)$, Π being the prime field of the rational numbers. An exact construction problem worded with respect to a basis B of a construction field, must run: to construct either *a* figure H, or *all* figures H, or *the* figure H, that satisfies certain prescribed conditions with regard to certain basis objects, and in the last case the problem is only significant if it has one unique solution .(The investigation in the structure of construction fields is closely related to the investigation in the possibility of solving exact problems; the data of an exact problem constitute the basis of the construction field in question, a line assumed constructed from two distinct points, a circle constructed from centre and circumference point).

Definition. Let be given a figure F, and let s_1, s_2, be a set of *F-equivalent points* constructed (from the objects of F) with the aid of the operations of Q; then the points s_1, s_2, will be called *geometrically conjugates over F (under Q)*. Analogous definitions may be given for geometrically conjugates of other kinds of objects or figures constructed from F.

Corollary. If an exact problem has been worded with respect to a basis B and if the problem has more solutions, these solutions are geometrically conjugate over B; conversely, if a problem worded with respect to B has solutions all geometrically conjugate over B, the problem is exact.

In the case that an operation P_4' (*"branching operation"*) has been applied to a circle and a line passing through the centre, we will often use the mere expression "geometrically conjugates", tacitly understanding that F will be the figure formed by circle and line.

Example 1. Exact construction problem: given a basis B consisting of two points, $z(0,0)$ and $e_y(1,0)$; to construct *a* point p that may get the coordinates $(1+\sqrt{2},0)$ in a suitable coordinate system (the coordinates of z and e_y considered fixed). (Fig. 7). Construction: $L_1(z,e_y) = \bar{y}$; $S_1[P_4'\{L_3(\text{inc } z, \perp \bar{y}), C_1(\dot{z},e_y)\}] = e_x(0,1)$; $S_1[P_4'\{\bar{y}, C_1(\dot{e}_y,e_x)\}] = \dot{p}(1+\sqrt{2},0)$.

There are two geometrically conjugates over B: if one of them is called $(1+\sqrt{2},0)$, then the other one will have coordinates $(1-\sqrt{2},0)$.

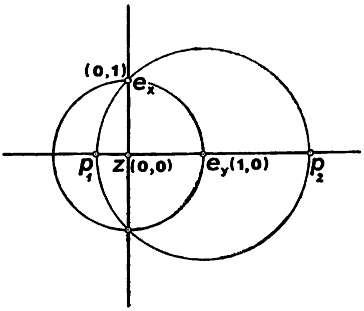

FIG. 7

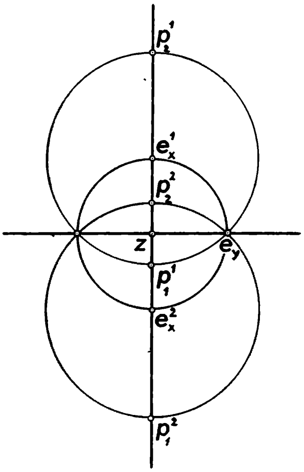

FIG. 8

Example 2. Exact construction problem: given a basis B consisting of two points $z(0,0)$, $e_y(1,0)$; to construct all geometrically conjugates over B of *a* point $p(0,1+\sqrt{2})$. (Fig. 8.) Construction:
$S_1\,[P_4'\,\{L_3(\text{inc}\,z,\ \perp\,\bar{y}),\,C_1(\dot{z},e_y)\}] = e_x^{\,i}(0,1),\ i=1,2;$
this notation indicates that a *reduplication* of the construction must follow; the next step in the construction chain must be carried out in both cases corresponding with the two possible selections of the point $e_x(0,1)$.
$S_1\,[P_4'\,\{L_3(\text{inc}\,z,\ \perp\,\bar{y}),\,C_1(\dot{e}_x^{\,i},e_y)\}] = p_k^{\,i},\ i=1,2;\ k=1,2.$

We find four geometrically conjugates over B; the coordinates of each of them are not determined uniquely *a priori*, because certain (arbitrary) selections for them can be made. These selections once made, each of them may get its own name; the points are all distinct.

In both cases two coordinate systems can be found with respect to which the points z and e_y have their prescribed coordinates. In the first example the requisite points are invariant under the permutations of these coordinate systems, in the second example they are not.

Now we will introduce some new notions in order to describe the possible construction processes.

Let be given a basis B and a construction chain K leading from B to a certain object, or to certain objects of $C_Q(B)$. K is finite so contains at most a finite number of branching operations P_4', each of them producing a couple of points simultaneously. After such a branching operation three different kinds of constructive continuations are possible.

Let W represent the figure consisting of B and of the set of objects of $C_Q(B)$ occurring already by the application of the operations of K until just before a certain branching operation P_4' of K, producing two points s_1 and s_2. Consider the orthogonal-linear construction field $C_L(W)$ upon the *basis* W Now the points s_1 and s_2 are either $C_L(W)$-equivalent or not. The question whether they are or not is decided by analytical means.

(*a*) *s_1 and s_2 are not $C_L(W)$ -equivalent.* This implies that, e.g., s_1 satisfies a certain decidable 0-relation with respect to certain objects, or a certain object, of $C_L(W)$, while s_2 does not satisfy it. (Decidable: with the aid of the decision operations available.) Let us ussume that each object of W has got a fixed "name" in coordinates, possibly arbitrarily fixed after branching operations. Then each object in $C_L(W)$ is constructible uniquely because the operations of L are univalent. Then, by using the available decision operations, the points s_1 and s_2 can be separated, so that they are constructible separately and uniquely with respect to $C_L(W)$. s_1 and s_2 are no geometrically conjugates over $C_L(W)$. The branching operation P_4' in question will be followed, in K, by a selection operation of the type: $S_2(s_1 \mathbin{\&} s_2) = s$ (or $= s_1$ *and* s_2) *so that the* 0*-relation in question exists between* s (*or* s_1) *and a certain object of* $C_L(W)$.

Example: reflection of a point a with respect to a point b: $S_2\ [P_4'\ \{L_1(a,b),\ C_1(\dot{b},a)\}] = c$ *so that* $c \neq a$ (*or*: $= c_1$ *and* c_2 *so that* $c_2 = a$).

(*b*) *s_1 and s_2 are $C_L(W)$-equivalent.* This implies: s_1 and s_2 are geometrically conjugates over W

If the operation P_4' that has produced the points s_1 and s_2, is followed, in K, by a selection operation S_1, whereby *one single* arbitrary selection suffices in order to finish the construction, such a selection will be called a *single selection.* Such selections will be written either $S_1(s_1 \mathbin{\&} s_2) = s$, or $S_1(s_1 \mathbin{\&} s_2) = s_1$ *and* s_2.

Example. Consider example 1 on page 65. The selection of e_x is a single selection.

If the operation P_4' in question is followed, in K, by a selection operation S_1, after which the next construction process represented by the chain K, must be continued in both possible directions, such a selection will be called a *double selection.* It is denoted, e.g., either by $S_1(s_1 \,\&\, s_2) = s_i$, $i = 1,2$, or by $S_1(s_1 \,\&\, s_2) = s_i \text{ and } s_j$, $i,j = 1,2$, $i \neq j$.

Example. Consider example 2 on page 66. Both selection operations are double selections. We remark that, if all selections occurring in a construction chain are double selections, all objects produced after the last step of the construction chain are geometrically conjugates over the basis. Conversely, if all geometrically conjugates are required, all branching operations occurring in the construction chain that cannot be followed by an operation S_2 (the case (*a*)) must be followed (in K) by double selections S_1.

Given a certain construction chain K containing a finite number of branching operations (P_4'). Consider only those that are followed by selection operations S_1 (Cases (*b*)). Forming another construction chain K' from K by considering all these selection operations as *double selections,* we contemplate the *figure* $F(K)$ consisting of B and all objects obtained by applying the operations of K' $F(K)$ will be called the *complete figure with respect* to K.

If K leads from B to some object of $C_Q(B)$, $F(K)$ contains in any case all geometrically conjugates (over B) of that object in question.

We may consider the automorphisms of a complete figure $F(K)$ that leave B pointwise fixed.

Example. Consider the exact construction problem mentioned before (Example 2, *p.* 66, fig. 8). The letters e_x^1 and e_x^2 are written down arbitrarily; so are the letters p_1^1, p_2^1, p_1^2, p_2^2, on the condition that p_1^1 and p_2^1 belong to the selected point e_x^1, and the others to e_x^2. If p_1^1 is called $(0,1+\sqrt{2})$, then p_2^1 is $(0,1-\sqrt{2})$; then p_2^2 must be $(0,-1-\sqrt{2})$ because $p_2^2,e_y \simeq p_1^1,e_y$, and finally we find $p_1^2 = (0,-1+\sqrt{2})$. The automorphisms of the figure $F(K)$ consisting of all points and lines appeared, constitute a *four group*; the elements are I, $A(p_1^1 \leftrightarrows p_2^1;\ p_2^2 \leftrightarrows p_1^2)$, $B(p_1^1 \leftrightarrows p_2^2;\ p_2^1 \leftrightarrows p_1^2)$, $C(p_1^1 \leftrightarrows p_1^2;\ p_2^1 \leftrightarrows p_2^2)$.

5. Adjunction operations.

The use of the *geometrical adjunction operation* A^g(Cf. p. 12) will be admitted in order to procure the basis points for an adjunctionless construction field $C_Q(B)$ and in order to construct objects in an adjunctive construction field $C_{Q(A)}(B)$ (Cf. p. 14).

We will prove that a basis containing two distinct points is *sufficient.*

Theorem 18. *Let* $z(0,0)$, $e_y(1,0)$, t_λ $(\alpha_\lambda,\beta_\lambda)$, $\lambda \in \Lambda$, *represent a basis* B *for a construction field* $C_Q(B)$, *and let* t (τ_1,τ_2) *be a point transcendentally adjoined to* B, *calling* $B' = [B,t]$. *If* p (π_1,π_2) *is a point of the construction field* $C_Q(B')$ *independent of the adjoined point* t, *it belongs already to* $C_Q(B)$. *So* B *is sufficient with respect to* Q.

Dem. Consider π_1. $\pi_1 \in \Omega(\alpha_\lambda,\beta_\lambda,\tau_1,\tau_2)$; π_1 is a root of an irreducible polynomial with coefficients in $\Pi(\alpha_\lambda,\beta_\lambda,\tau_1,\tau_2)$, Π being the prime field of the rational numbers. p is independent of the adjoined point; this means that the coefficients in the irreducible polynomial (with "highest" coefficient 1) for π_1 must be independent of τ_1 and τ_2. Just as shown in the demonstration of theorem 1 this implies, that these coefficients must belong to $\Pi(\alpha_\lambda, \beta_\lambda)$. Consequently $\pi_1 \in \Omega(\alpha_\lambda,\beta_\lambda)$; analogously we may derive: $\pi_2 \in \Omega(\alpha_\lambda,\beta_\lambda)$, proving the theorem.

In the sequel it is assumed that a basis containing two distinct points is given; with respect to the operation set Q the adjunction operations will no more be used. Beside the construction fields $C_Q(B)$ we shall, however, often consider the orthogonal-linear construction fields $C_L(B)$; then, we shall see, that it is convenient to use certain determinate adjunction operations concerning the operation set L, this being a subset of Q. (The only branching operation in Q is P_4'; this operation is the only one that distinguishes Q from L.).

Let P_4' be applied to a line and a circle of an orthogonal-linear construction field $C_L(B)$ producing the points s_1 & s_2. If these points are geometrically conjugate over $C_L(B)$, they do not belong to $C_L(B)$ (Cf. section 4, pp. 66 ff.). In theorem 19 the inverse of this statement will be proved: if two points s_1 & s_2 produced by the application of P_4' to a line and a circle of an orthogonal-linear construction field $C_L(B)$ do not belong to $C_L(B)$, they are $C_L(B)$-equivalent and consequently geometrically conjugate over $C_L(B)$. (It is clear that if one of these points s_1 & s_2, say s_1, does not belong to $C_L(B)$, s_2 is no more a point of $C_L(B)$, because reflection with respect to the centre of the circle is possible in $C_L(B)$.)

Definition. A branching operation applied to objects of an orthogonal-linear construction field $C_L(B)$ and producing two points outside $C_L(B)$, is called an *extending branching operation.*

Let B be a basis in C; let an extending branching operation P_4' be applied to a line and a circle of $C_L(B)$, producing two points s_1 & s_2 outside $C_L(B)$; form a new basis $B' = [B,s_1]$ by adding one of the points s_1 & s_2 to B; then we will say that B is extended to B' by the application of a determinate adjunction operation A^d to B and s_1 (in the construction geometry (C,L)).

6. Automorphisms of construction fields.

Let $z(0,0)$, $e_y(1,0)$, $t_\lambda(\alpha_\lambda,\beta_\lambda)$, $\lambda \in \Lambda$, constitute a basis B (containing two distinct points) for the construction field $C_Q(B)$ as well as for the orthogonal-linear construction field $C_L(B)$; let a branching operation P_4' be applied to a line and a circle of $C_L(B)$ producing the points s_1 & s_2. Finally let one of these points, say s_1, not belong to $C_L(B)$ (so neither does s_2). The adjunction of s_1 to B gives $B' = [B,s_1]$. Next we have:

Theorem 19. *The points s_1 & s_2 are $C_L(B)$-equivalent. The permutations of the points s_1 & s_2 give rise to automorphisms of $C_L(B')$ which leave $C_L(B)$ objectwise fixed. These automorphisms constitute a two-cyclic group. The only elementary objects (points, lines, circles) of $C_L(B')$ invariant under these automorphisms are those that belong to $C_L(B)$ already.*

In the course of the demonstration we find a representation for these automorphisms in the various possible cases.

Dem. We remind the reader of the difference between P-field and Γ-field. (Chapter IV, p. 44) The orthogonal-linear construction field is at least a Γ-field; it is also a P-field if and only if it contains a point $e_x(0,1)$. The place of *the* point $(0,1)$ is not determined (uniquely); in analytical language: the transformation from the absolute coordinate system to a coordinate system with respect to which the given basis points have their coordinates as described above, is not determined (uniquely). If the orthogonal-linear construction field $C_L(B)$ is a Γ-field, the including P-field admits of two automorphisms leaving the Γ-field objectwise fixed (Cf. Theorem 15, p. 57). $C_L(B)$ itself does not admit of automorphisms that leave B objectwise fixed, but identity.

The theorem will be proved in two cases: for a P-field and for a Γ-field; the latter case again is split up into two parts; the first part corresponds with the case of a P-field; nevertheless we prefer a separate treatment, because we shall need to show how the different cases occur.

(*a*) *B is such a basis that* $C_L(B)$ *is a P-field.*

The P-field is constructively closed with respect to the operations of L (Cf. Chapter IV, theorem 8). But when a P-circle and a P-line passing through the centre of the circle intersect (operation P_4'), the coordinates of the points of intersection $s_{1,2}$ may be represented by $x_{s_{1,2}} = \varkappa \pm \lambda\sqrt{\vartheta}$, $y_{s_{1,2}} = \mu \pm \nu\sqrt{\vartheta}$, where $\vartheta = (1+\vartheta_0{}^2)$, $\varkappa$, λ, μ, ν, ϑ_0 being elements of P, and taking the sign $+$ for one of these points and the sign $-$ for the other one. We have $\sqrt{\vartheta} \notin P$; we find:

(*i*) neither s_1 nor s_2 is coincident with any P-point (D^{pp}),

(*ii*) both s_1 and s_2 are incident with only one and the same P-line, viz. the line from which they have been constructed, otherwise they should be P-points themselves, contrary to the assumption (D^{pl}),

(*iii*) when s_1 is incident, respectively excident, with a P-circle, then it will also be the case with $s_2(D^{pc})$; substituting the coordinates of s_1 & s_2 into the equation of the P-circle in question we reduce the proof of the last statement to the well known property: if $\sqrt{\vartheta} \notin P$, and $\alpha,\beta \in P$, and $\vartheta \in P$, the equalities $\alpha + \beta\sqrt{\vartheta} = 0$ and $\alpha - \beta\sqrt{\vartheta} = 0$ follow one from the other.

So the points s_1 & s_2 are $C_L(B)$-equivalent.

The adjunction of the coordinates of s_1 (and s_2) to P represents an algebraical extension of P to $P(\sqrt{\vartheta})$ by the root of the quadratic equation $x^2 - \vartheta = 0$, which is irreducible in P.

The orthogonal-linear construction field $C_L(B')$ upon $B' = [B,s_1]$ is *the set of all* $P(\sqrt{\vartheta})$*-objects*; it is the $P(\sqrt{\vartheta})$-field with respect to the same coordinate system as was used for the original P-field. The permutation of the points s_1 & s_2 generates a *permutation of algebraicly conjugate* $P(\sqrt{\vartheta})$*-objects*; this permutation is reflected by the substitution of $-\sqrt{\vartheta}$ for $\sqrt{\vartheta}$ in the coordinates of the $P(\sqrt{\vartheta})$-points and in the equation coefficients of the $P(\sqrt{\vartheta})$-lines and circles. Moreover, any relation of the first category (0-relation) existing between two $P(\sqrt{\vartheta})$-objects similarly exists between their algebraicly conjugate objects, so that these relations are invariant under the permutations of the points s_1 & s_2.

This means that such a permutation generates an automorphism of $C_L(B')$ *that leaves* $C_L(B)$ *objectwise fixed.*

Together with identity this automorphism constitutes a *two-*

cyclic automorphism group, isomorphic with the Galois group of $P(\sqrt{\vartheta})$ *over* P.

The $P(\sqrt{\vartheta})$-objects that are invariant under the automorphisms of this group are those and only those which belong to the original P-field.

(*b*) *B is such a basis that* $C_L(B)$ *is a Γ-field.*

The Γ-field is constructively closed with respect to the operations of L (Cf. Chapter IV, theorem 8). But when a Γ-circle an a Γ-line passing through the centre of the circle intersect (operation P_4'), the coordinates of the points of intersection by $x_{s_{1,2}} = \mu \pm \nu \sqrt{\vartheta}$, $y_{s_{1,2}} = \mu \pm \nu \sqrt{\vartheta}$, where $\vartheta = (1 + \vartheta_0^2)$, $\varkappa$, λ, μ, ν, ϑ_0 being elements of P, and taking the sigh $+$ for one of these points and the sign $-$ for the other one.

Now there are two possibilities either $\sqrt{\vartheta} \notin P$ *or* $\sqrt{\vartheta} \in P$.

First case: $\sqrt{\vartheta} \notin P$.

In the same way as we did in (*a*) we can show that both conjugates are $C_L(B)$-equivalent, that the adjunction of s_1 (and s_2) to B corresponds with an algebraic extension of P to $P(\sqrt{\vartheta})$ of the second degree, that the permutations of s_1 & s_2 induce automorphisms of $C_L(B')$ which leave $C_L(B)$ objectwise fixed, and that these automorphisms correspond with the field automorphisms of $P(\sqrt{\vartheta})$ over P, where P is the field belonging to $C_L(B)$.

Furthermore we shall show, that, in this case, $C_L(B')$ *generally is a* Γ'-field, which shall belong to the $P(\sqrt{\vartheta})$-field mentioned under (*a*).

The "first case" always occurs when P_4' *is applied to a Γ-circle and a Γ-line of the type* $\bar{r}(y = \varrho_1 x + \varrho_2)$.

The parallel-translation is performable in the Γ-field; so without loss of generality we may suppose that the branching operation producing the points s_1 & s_2 meant in the theorem, is applied to a Γ-circle with centre $z(0,0)$ and the equation $x^2 + y^2 + \tau_3 = 0$, τ_3 being an even expression in the β_λ, and a line passing through z, with the equation $y = \varrho_1 x$, ϱ_1 being an odd expression in the β_λ (with coefficients in $\Pi(\alpha_\lambda)$).

Then $x_{s_{1,2}} = \pm \sqrt{-\tau_3/(1+\varrho_1^2)}$ is an even expression in the β_λ and consequently $y_{s_{1,2}}$ is an odd expression in the β_λ. In this case $\vartheta =$

$-\tau_3/(1+\varrho_1^2)$; $\sqrt{\vartheta} \in P$·would imply s_1 & $s_2 \in C_L(B)$ which is contrary to the hypothesis of the theorem. Consequently $\sqrt{\vartheta} \notin P$.

In order to characterize the construction field $C_L(B')$ we note that it is the orthogonal-linear construction field upon B' consisting of z, e_y, t_λ, $s_1(\sqrt{\vartheta}, \varrho_1\sqrt{\vartheta})$, where $\vartheta = (1 + \vartheta_o^2)$, ϱ_1, ϑ_o being odd expressions in the β_λ. *We see that* $\sqrt{\vartheta}$ *occurs as a new magnitude* a_0. Consider the fields Π; $\Delta = \Pi(\alpha_\lambda)$; $\Delta(\sqrt{\vartheta}) = \Pi(\alpha_\lambda, \sqrt{\vartheta})$; $P(\sqrt{\vartheta}) = \Pi(\alpha_\lambda, \sqrt{\vartheta}, \beta_\lambda)$. Now we find: *A point* $p(\pi_1,\pi_2)$ *belongs to* $C_L(B')$ *if and only if* $\pi_1, \pi_2 \in P(\sqrt{\vartheta})$, π_1 *being an even and* π_2 *being an odd rational expression in the* β_λ *with coefficients in* $\Pi(\alpha_\lambda, \sqrt{\vartheta})$.

So $C_L(B')$ is the $\Gamma(\sqrt{\vartheta})$-field with respect to the subfield $\Pi(\alpha_\lambda, \sqrt{\vartheta})$ of P. According to theorem 9 this $\Gamma(\sqrt{\vartheta})$-field may be the $P(\sqrt{\vartheta})$-field, if B has a special structure: this is the case if and only if there exists an algebraic equality $\varphi'(\beta_\lambda) = \varphi''(\beta_\lambda) \neq 0$, φ' being an even and φ'' being an odd rational expression in the β_λ together, with coefficients in $\Pi(\alpha_\lambda, \sqrt{\vartheta})$. Because $\sqrt{\vartheta}$ is an even expression in the β_λ, the following wording will be used for the condition: the $\Gamma(\sqrt{\vartheta})$-field is the $P(\sqrt{\vartheta})$-field if and only if the field $P(\sqrt{\vartheta})$ contains an element $\varphi \neq 0$ that is simultaneously an even and an odd function of the β_λ.

The "first case" *may* occur *when* P_4' *is applied to a* Γ*-circle and a* Γ*-line of the type* $\bar{s}(x = \sigma)$.

As we have done just now we again choose a Γ-circle with centre $z(0,0)$ and we take its points of intersection with the y-axis, s_1,s_2. Let $d(\delta_1,\delta_2)$ be any Γ-point, then any Γ-circle with centre z can be represented by the equation $x^2 + y^2 = \delta_1^2+\delta_2^2$. $\delta_1 \neq 0$, otherwise s_1,s_2 should be Γ-points themselves.

Now $x_{s_{1,2}} = 0$, $y_{s_{1,2}} = \pm |\delta_1| \sqrt{1 + (\delta_2/\delta_1)^2}$, this being an even expression in the β_λ. $\vartheta_0 = \delta_2/\delta_1$ is always an odd expression in the β_λ.

$C_L(B')$ is the orthogonal-linear construction field upon B', consisting of z, e_y, t_λ, $s_1(0, |\delta_1| \sqrt{1+\vartheta_o^2})$. Here we see that $\sqrt{1+\vartheta_o^2} = \sqrt{\vartheta}$ *must be considered as a new magnitude* β_0, Consider the fields Π; $\Delta = \Pi(\alpha_\lambda)$; $P = \Pi(\alpha_\lambda, \beta_\lambda)$, $P(\sqrt{\vartheta}) = \Pi(\alpha_\lambda, \beta_\lambda, \sqrt{\vartheta})$. Then we find: *A point* $p(\pi_1,\pi_2)$ *belongs to* $C_L(B')$ *if and only if* $\pi_1, \pi_2 \in P(\sqrt{\vartheta})$, π_1 *being an even and* π_2 *being an odd rational expression in the* β_λ *and* $\sqrt{\vartheta}$, *with coefficients in* $\Pi(\alpha_\lambda)$.

So in this case $C_L(B')$ is the $\Gamma(\sqrt{\vartheta})$–field with respect to the subfield $\Pi(\alpha_\lambda)$ of P.

According to theorem 9 this $\Gamma(\sqrt{\vartheta})$-field may be the $P(\sqrt{\vartheta})$-field, if B has a special structure: this is the case if and only there exists an algebraic equality $\varphi'(\beta_\lambda, \sqrt{\vartheta}) = \varphi''(\beta_\lambda, \sqrt{\vartheta}) \neq 0$, φ' being an even and φ'' being an odd rational expression in the β_λ and $\sqrt{\vartheta}$ together, with coefficients in $\Pi(\alpha_\lambda)$. Because $\sqrt{\vartheta}$ is an even expression in the β_λ, the following wording will be used for the condition: the $\Gamma(\sqrt{\vartheta})$-field is the $P(\sqrt{\vartheta})$-field if and only if the field $P(\sqrt{\vartheta})$ contains an element $\neq 0$ that is simultaneously an even and an odd function of the β_λ and $\sqrt{\vartheta}$ together.

Second case: $\sqrt{\vartheta} \in P$.

As has been shown above this case can only occur when P_4' is applied to a Γ-circle and to a Γ-line of the type $\bar{s}\,(x = \sigma)$.

Again we use a Γ-circle with centre z, without loss of generality. The points of intersection $s_{1,2}$ of $x^2+y^2+\tau_3 = 0$ and $x = 0$ can be represented by $x_{s_{1,2}} = 0$, $y_{s_{1,2}} = \pm \sqrt{\vartheta}$, ϑ being an even expression in the β_λ, because τ_3 is even in the β_λ. Moreover, we have $\sqrt{\vartheta} \in P$; we find:

(*i*) neither s_1 nor s_2 is coincident with any Γ-point (D^{pp});

(*ii*) both s_1 and s_2 are incident only with one and the same Γ-line, viz. the line from which they have been constructed, otherwise they should be Γ-points themselves (D^{ll});

(*iii*) if s_1 is incident, respectively excident, with a Γ-circle, then this will also be the case with s_2 (D^{pc}): substituting the coordinates of $s_1(0, \sqrt{\vartheta})$ into the equation $x^2+y^2+\gamma_1 x+\gamma_2 y+\gamma_3 = 0$ of a Γ-circle (so that γ_1 and γ_3 are even expressions and γ_2 is an odd expression in the β_λ) we find:
$\vartheta + \gamma_3 = -\gamma_2 \sqrt{\vartheta}$. The left hand side of this equality is an even expression in the β_λ; this must also be the case with the right hand side, which implies $\gamma_2 = 0$; consequently $s_2(0, -\sqrt{\vartheta})$ is also incident with this circle.

So the points s_1 & s_2 are $C_L(B)$-equivalent.

By the adjunction of s_1 (and s_2) to B *the construction field $C_L(B)$ will be extended to the P-field including our Γ-field.*

For let the point $u(0, \sqrt{\vartheta})$ be selected (arbitrarily) from s_1 & s_2: $S_1(s_1 \,\&\, s_2) = u$. The point $u'(\sqrt{\vartheta}, 0)$ is a Γ-point. Then $e_x(0,1)$ can be constructed:

$$L_1\,[P_1'\,\{L_3\,(\text{inc}\,u',\ \perp \bar{y}),\, L_2\,(\text{inc}\,u,\ /\!/\ \bar{y})\},\, z] = \bar{n};$$
$$P_1' < L_2\,[\text{inc}\,P_1'\,\{L_3(\text{inc}\,e_y,\ \perp \bar{y}), \bar{n}\},\ /\!/\ \bar{y}],\ \bar{x} > = e_x(0,1).$$

We know that the orthogonal-linear construction field upon a basis containing the points $z(0,0)$, $e_y(1,0)$, $e_x(0,1)$, is the linear construction field upon that basis; this is a P-field. Because $C_L(B)$ is a Γ-field (Cf. theorem 9, p. 45), every element ε of P may be uniquely written $\varepsilon = \varepsilon' + \varepsilon''$, ε' being an even and ε'' being an odd rational expression in the β_λ with coefficients in $\Pi(\alpha_\lambda)$.

The permutation of the points s_1 *&* s_2 *generates a permutation of the P-objects of the construction field* $C_L(B')$*, that may be called geometrically conjugates over the underlying* Γ*-field:*

the points $p_1(\pi_1' + \pi_1'', \pi_2' + \pi_2'')$ and $p_2(\pi_1' - \pi_1'', -\pi_2' + \pi_2'')$, the lines $\bar{r}_1(y = (\varrho_1' + \varrho_1'')x + \varrho_2' + \varrho_2'')$ and $\bar{r}_2(y = (-\varrho_1' + \varrho_1'')x - \varrho_2' + \varrho_2'')$, the lines $\bar{s}_1(x = \sigma' + \sigma'')$ and $\bar{s}_2(x = \sigma' - \sigma'')$, the circles $\bar{\bar{t}}_1(x^2 + y^2 + (\tau_1' + \tau_1'')\, x + (\tau_2' + \tau_2'')\, y + \tau_3' + \tau_3'' = 0)$ and $\bar{\bar{t}}_2(x^2 + y^2 + (\tau_1' - \tau_1'')\, x + (-\tau_2' + \tau_2'')\, y + \tau_3' - \tau_3'' = 0)$ will be interchanged.

Moreover every relation of the first category (0-relation) existing between P-objects similarly exists between their geometrically conjugate objects, so that these relations are invariant under the permutations of the points s_1 & s_2. *This means that these permutations generate automorphisms of* $C_L(B')$ *that leave* $C_L(B)$ *objectwise fixed.*

These automorphisms constitute a two-cyclic automorphism group. The P-objects being invariant under the automorphisms of this group are exactly the original Γ-cbjects. So theorem 19 has been proved.

Remark. In the case that the adjunction of two conjugate points to a Γ-field leads to the including P-field (Second case of the demonstration of theorem 19), the automorphisms of the latter that leave the Γ-field objectwise fixed, correspond with the permutations of the isotropic points i_1 & i_2. (Cf. Remark 2, p. 55, Chapter IV).

Let B_0 be a basis in C consisting of the points $z(0,0)$, $e_y\,(1,0)$, $t_\lambda(\alpha_\lambda, \beta_\lambda)$, $\lambda \in \Lambda$, Λ some index system. We form the orthogonal linear construction field $C_L(B_0)$ and extend B_0 to B_1 by the determinate adjunction of one of the two $C_L(B_0)$-equivalent points produced by the application of P_4' to a line and a circle of $C_L(B_0)$. We form $C_L(B_1)$ and extend B_1 to B_2 by the adjunction of one of the $C_L(B_1)$-equivalent points produced by the application of P_4' to a line and a circle of $C_L(B_1)$.

The automorphisms of $C_L(B_1)$ *which leave* $C_L(B_0)$ *objectwise fixed* *can be continued to automorphisms of $C_L(B_2)$ which leave $C_L(B_0)$ objectwise fixed.*

This process can be extended to the case where a basis B' arises by an enumerable number of successive adjunctions of the kind mentioned. Let B' consist of the points $z(0,0)$, e_y $(1,0)$, $t_\lambda(\alpha_\lambda, \beta_\lambda)$, $\lambda \in \Lambda$, $s_\mu(\alpha_\mu, \beta_\mu)$, $\mu = 1,2, \quad m$, now the orthogonal-linear construction field $C_L(B_0)$ is either the P_0-field ($p(\pi_1,\pi_2) \in C_L(B_0)$ if and only if $\pi_1, \pi_2 \in P_0 = \Pi(\alpha_\lambda,\beta_\lambda)$ or the Γ_0-field ($p(\pi_1,\pi_2) \in C_L(B_0)$ if and only if $\pi_1, \pi_2 \in P_0$ but π_1 being an even and π_2 being an odd rational expression in the β_λ with coefficients in $\Pi(\alpha_\lambda)$)).

Moreover, as a consequence of theorem 19 we find: the orthogonal-linear construction field $C_L(B')$ is either the P'-field ($p(\pi_1, \pi_2)$ $\in C_L(B')$, if and only if $\pi_1, \pi_2 \in P' = \Pi(\alpha_\lambda, \beta_\lambda, \alpha_\mu, \beta_\mu)$), or the Γ'-field ($p(\pi_1,\pi_2) \in C_L(B')$, if and only if $\pi_1, \pi_2 \in P'$, but π_1 being an even and π_2 being an odd rational expression in the β_λ, β_μ with coefficients in $\Pi(\alpha_\lambda, \alpha_\mu)$).

We consider the series of bases $B_0 \subset B_1 \subset B_2 \subset \quad \subset B' = B_m$, and simultaneously the corresponding series of orthogonal-linear construction fields $C_L(B_0) \subset C_L(B_1) \subset C_L(B_2) \subset \quad \subset C_L(B')$. There are four cases.

I. B_0 is such a basis that $C_L(B_0)$ is the P_0-field; a special property for B_0 is required; then $C_L(B')$ is the P'-field.

II. B_0 is such that $C_L(B_0)$ is the Γ_0-field, while $C_L(B')$ is the Γ'-field; no special property for B_0 required, B_0 is a "general" basis, i.e., B_0 has been formed by successive (geometrically) transcendental adjunctions.

III. B_0 is such that $C_L(B_0)$ is the Γ_0-field, while $C_L(B')$ is the P'-field, whereby the algebraic degree of P' over P_0 amounts 2^m, m being the number of determinate adjunctions performed to reach B' A special property for B_0 is then required: P' must contain an element $\varphi \neq 0$ that is simultaneously an even and an odd function of the β_λ.

Consider, for these three cases, the series of fields $P_0 \subset P_1 \subset P_2 \subset \quad \subset P'$ The automorphisms of $C_L(B')$ that leave $C_L(B_0)$ objectwise fixed correspond one-to-one with the automorphisms of the field P' over P_0. *The automorphism group of $C_L(B')$ over $C_L(B_0)$ is isomorphic with the Galois group of P' over P_0.*

IV B_0 is such a basis that $C_L(B_0)$ is the Γ_0-field, while $C_L(B')$ is the P'-field, whereby the algebraic degree of P' over P_0 amounts

2^{m-1}, m being the number of determinate adjunction operations performed to reach B' This means that one of the extensions of a basis B_i to the basis B_{i+1} corresponds with the transition of a Γ_i-field to the including P_i-field. One of the symbols $\subset$ in the series of fields $P_0 \subset P_1 \subset P_2 \subset \quad \subset P'$ must be replaced by $=$ There are no special conditions for B_0, B_0 is a "general" basis.

We will show: *the automorphism group* $\mathfrak{G}$ *of* $C_L(B')$ *over* $C_L(B_0)$ *is isomorphic with the direct product of the Galois group* $\mathfrak{R}$ *of* P' *over* P_0 *and a two-cyclic permutation group* $\mathfrak{P}$ (corresponding with the permutation group of the isotropic points i_1 & i_2).

Proof. Because P' does not contain any element $\varphi \neq 0$ that is simultaneously an even and an odd function of the β_λ, any element ε of P' can be uniquely written $\varepsilon' + \varepsilon''$ being an even and ε being an odd function of the β_λ. ε' is a root of an irreducible polynomial with coefficients in $\Pi(\alpha_\lambda,\beta_\lambda)$, these coefficients being even rational expressions in the β_λ with coefficients in $\Pi(\alpha_\lambda)$; all algebraic conjugates of ε' in P' over $P_0 = \Pi(\alpha_\lambda, \beta_\lambda)$ are similarly even functions of the β_λ. Analogously: all algebraic conjugates of ε'' in P' over P_0 are odd functions of the β_λ.

Let $p(\pi_1, \pi_2)$ be a point of $C_L(B')$; we write $\pi_1 = \pi_1' + \pi_1''$, $\pi_2 = \pi_2' + \pi_2''$, π_1' and π_2' being even, π_1'' and π_2'' being odd functions of the β_λ; the image of p under an automorphism $\mathfrak{g}$ out of the group $\mathfrak{G}$ is represented by r $(\varrho_1' + \varrho_1'' \quad \varrho_2'+\varrho_2'')$. Because the permutations of β_λ and $-\beta_\lambda$ (for all $\lambda \in \Lambda$) map even functions of the β_λ onto even functions, and odd functions of the β_λ onto odd, and because the representation $\varepsilon = \varepsilon' + \varepsilon''$ is unique, any element of $\mathfrak{G}$ can be written uniquely as the product of an element of $\mathfrak{R}$ and an element of $\mathfrak{P}$; moreover, any element of $\mathfrak{R}$ is interchangeable with any element of $\mathfrak{P}$, proving the statement. Conclusion:

Theorem 20. *Let* $z(0,0)$, $e_y(1,0)$, t_λ $(\alpha_\lambda,\beta_\lambda)$, $\lambda \in \Lambda$, *constitute a basis* B *in the construction geometry* (C, Q) *and let* B' *be a set of points in the adjunctionless construction field* $C_Q(B)$ *containing* B; *let* P *represent the coordinate field of the orthogonal-linear construction field* $C_L(B)$, *and* P' *the coordinate field of the orthogonal-linear construction field* $C_L(B')$; *then the group of automorphisms of* $C_L(B')$ *that leave* $C_L(B)$ *objectwise fixed (the automorphism group of* $C_L(B')$ *over* $C_L(B)$*) is either isomorphic with the Galois group of* P' *over* P, *or isomorphic with the direct product of this group and a two-cyclic group.*

7. Objects in a C_Q-field being constructible uniquely.

Already some fifty years ago (1909) H. Tietze [1] strongly objected against the current conception of constructibility. Wishing to describe the construction process of a certain requisite object from a set of given objects (the basis) by means of unique operations, Tietze was not content to find the objects required among a set of conjugate ones and to select them from this set with the aid of "unmathematical" means, so as, e.g., by looking at some drawn figure, etc. In the next chapter we shall see that it is possible to introduce certain decision operations solving the problem, and being by no means "unmathematical" Nevertheless, to that end the introduction of the concept of *order* into geometry is necessary.

In the present congruence geometry order has not been introduced. Tietze determined the set of points, lines, and circles, the construction of which is independent of the arbitrary selections made after the application of extending branching operations.

In fact Tietze's investigations were concerned with the ordinary Euclidean construction geometry, but that does not necessarily make any difference. (Cf. Chapter VII, p. 91).

Definition. *When B is a basis in the construction geometry (C, Q) the point p is constructible uniquely in the construction field $C_Q(B)$ if and only if there exists a construction chain K leading from B to p in such a manner that p is independent of the selections S_1 occurring in K.* (Similar definitions may be given for other kinds of objects of $C_Q(B)$.)

The subset of uniquely constructible objects of $C_Q(B)$ is denoted by $C^u{}_Q(B)$.

Corollary. When B is a basis in (C, Q) the point p is constructible uniquely in the construction field $C_Q(B)$ if and only if there exists a construction chain K leading from B to p in such a manner that p is *invariant under the automorphisms of the complete figure $F(K)$ belonging to K that leave B pointwise fixed.*

The following theorem is partially due to Tietze; in fact Tietze only considered such bases whereby the orthogonal-linear construction fields upon them were Γ-fields, though the discrimination between P-fields and Γ-fields was not made by him.

Theorem 21. *When $z(0,0)$, $e_y(1,0)$, $t_\lambda(\alpha_\lambda,\beta_\lambda)$, $\lambda \in \Lambda$, constitute a basis B of a construction field $C_Q(B)$, a point, a line, or a circle of $C_Q(B)$*

belongs to $C^u{}_Q(B)$ *if and only if the object in question belongs to the orthogonal-linear construction field* $C_L(B)$.

Dem. *Sufficiency.*

We have $L \subset Q$; L consists of univalent and consequently unique operations. So $C_L(B) \subseteq C^u{}_Q(B)$.

Necessity.

There exists a construction chain K leading from B to a certain point, a line, or a circle, say a point p for the sake of convenience, containing a finite number of extending branching operations. Let B' consist of the points occurring in the complete figure $F(K)$ belonging to K. $p \in C^u{}_Q(B)$, so p is invariant under all automorphisms of $F(K)$ that leave B pointwise fixed. These automorphisms induce uniquely all automorphisms of $C_L(B')$ that leave $C_L(B)$ objectwise fixed, because in $C_L(B')$ all objects are constructible uniquely by using univalent operations. The only points, lines, and circles, that remain fixed under these automorphisms are those of $C_L(B)$. Consequently $p \in C_L(B)$, proving the theorem.

Corollary. $C^u{}_Q(B)$ consists of those objects of $C_Q(B)$ that are invariant under all automorphisms of $C_Q(B)$ that leave B pointwise fixed.

Remark 1. It may occur that certain objects of $C^u{}_Q(B)$ do not belong to $C_L(B)$. However, these cannot be points, lines, or circles. We give an example: let us assume that $C_L(B)$ is a Γ-field. Although the points $e_x(0,1)$ and $d_x(0,-1)$ are not constructible separately and uniquely, the *square* upon the points $e_y(1,0)$, e_x, $d_y(-1,0)$, d_x is.

Although the equilateral triangle upon $z(0,0)$, e_y, $c_1(\frac{1}{2},\frac{1}{2}\sqrt{3})$ is not constructible uniquely, the *rhomb* upon the points z, $c_2(\frac{1}{2},-\frac{1}{2}\sqrt{3})$, e_y, c_1 is. Neither the square nor the rhomb belong to $C_L(B)$.

Remark 2. Concerning the possible solutions of an exact construction problem with respect to a certain basis we mention: any figure in a construction field $C_Q(B)$ that can be described uniquely in geometrical or analytical terms, is also constructible uniquely, and conversely. This is a result of the entire chapter.

Remark 3. With regard to a *limited* use of the operation P_4' in (C, Q) we find: if it is allowed to use P_4' only *once* and if the orthogonal-linear construction field upon a certain basis B is a Γ-field, we may obtain either a Γ'-field, being a proper quadratic extension of the former, or the including P-field. The latter case can only occur if we apply P_4' to a circle and a line normal to the x-axis.

8. Orientation. Rotation in (C,Q).

Because any construction field $C_Q^{\blacktriangledown}(B)$ upon a basis B containing two distinct points also contains a P-field, similar and opposite orientation can be defined and decided (in congruence geometry) for triangles from which (possibly using the remaining data) an isosceles right-angled triangle can be constructed uniquely. (Cf. Chapter IV, 6, p. 51).

Moreover, similar and opposite orientation can be defined and decided for *similar* (*and congruent*) *triangles*. We shall not go into all details. We only show that both *rotations* of a triangle a,b,c around the vertex a are constructible, if defined by the condition that the image b' of b is incident with a prescribed line $\bar{a}$ passing through a (fig. 9). If b inc $\bar{a}$, the construction is trivial. Assume b exc $\bar{a}$.

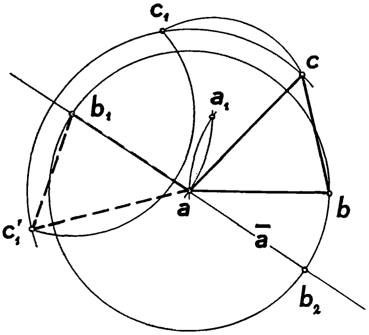

FIG. 9

$S_1\ [P_4'\ \{\bar{a},\ C_1(\dot{a},b)\}] = b_i',\ i = 1,\ 2;\ Refl_{L_1(b,b_i')}(a) = a_i;$ $Refl_{L_1(a,a_i)}(c) = c_i;\ Refl_{\bar{a}}\ (c_i) = c_i',\ i = 1, 2,$ these being the images of c under both rotations required.

Parallel translation is constructible in (C, Q), because it is constructible in reflection geometry already.

9. The number of geometrically conjugates of a point over a basis. A remark on a theorem of HILBERT.

A geometry satisfying the axioms of the first four groups of the

HILBERT axiom system (Cf. Grundlagen der Geometrie) will be called a HILBERT geometry. Congruence geometry is a HILBERT geometry without the notion of order.

The distinction between congruence and HILBERT construction geometry will be the extension of the operation set Q to Q^o, by adding a decision operation concerning betweenness. (Chapter VI.)

HILBERT stated a theorem on the number of geometrically conjugates of a point over a basis, whereby he did not take into account relations on order. So in fact his theorem is meant for congruence geometry.

According to the theorem of HILBERT ([1], § 39) a point p in an adjunctionless construction field $C_Q(B)$ upon a basis containing at least two distinct points, has 2^n (real) geometrically conjugates over B if n is the smallest number of square roots that suffices to compute the coordinates π_1, π_2 of p.

These square roots will have the form $\sqrt{\alpha^2+\beta^2+\quad}$ (a finite number of squares), whereby α, β, are parameters available in constructed shape, say as points ($a(\alpha,0)$, $b(\beta,0)$, or $a'(0,\alpha)$, $b'(0,\beta)$,

Consider a basis B_0, the points of B_0 represented by $z(0,0)$, $e_y(1,0)$, $t_\lambda(\alpha_\lambda, \beta_\lambda)$, $\lambda \in \Lambda$.

On the basis of certain construction problems we will show that the HILBERT theorem in its rigid form is not quite correct.

1. B_0 is such a basis that the orthogonal-linear construction field $C_L(B_0)$ is the P_0-field, where $P_0 = \Pi(\alpha_\lambda, \beta_\lambda; \lambda \in \Lambda)$; a point $e_x(0,1)$ has been constructed. Exact construction problem: to construct all geometrically conjugates over B_0 (under the operations of Q) of a point $p(\pi_1, \pi_2)$, π_1, π_2 being elements of $\Omega(\alpha_\lambda, \beta_\lambda)$, the coordinates taken with respect to the given coordinate system.

All rational operations are possible in the P_0-field; for each new square root operation an extending branching operation must be used; all extensions are algebraic; if n is the smallest number of sufficient square root operations, 2^n is the number of geometrically conjugates.

2. B_0 is such a basis that the orthogonal-linear construction field $C_L(B_0)$ is the P_0-field. Problem: to construct all points that may be geometrically conjugate with a point p having coordinates (π_1, π_2) with respect to any of the two possible coordinate systems. If n is

the smallest number of sufficient square root operations to compute π_1 and π_2, we find 2^{n+1} *solutions*; because both possible points $e_x(0,1)$ are *not* geometrically conjugate over B, each solution has 2^n geometrically *conjugates* over B. The problem is not exact.

3. B_0 is such a basis that the orthogonal-linear construction field $C_L(B_0)$ is the Γ_0-field. *Exact construction problem*: to construct all points that may be geometrically conjugate over B_0 with a point p having coordinates (π_1, π_2) with respect to any of the two possible coordinate systems. Both possible points $e_x(0,1)$ are geometrically conjugate over B. Let K be a construction chain leading from B_0 to all points required, K containing the smallest number k of extending branching operations sufficient to construct one of these points p. The complete figure $F(K)$ belonging to K delivers all geometrically conjugates. Every time we use an extending branching operation one point of each couple produced is adjoined to the basis of the preceding orthogonal-linear construction field, by applying the determinate adjunction operation A^d. Consider the series of bases $B_0 \subset B_1 \subset B_2 \subset \quad \subset B_k$, together with the corresponding series of orthogonal-linear construction fields $C_L(B_0) \subset C_L(B_1) \subset C_L(B_2) \subset \quad \subset C_L(B_k)$. The number of geometrically conjugates of p over B_0 amounts 2^k.

According to the results of theorem 19 we have (Cf. also p. 76): (the coordinate field belonging to $C_L(B_i)$ will be denoted by P_i) if $C_L(B_k)$ is the Γ_k-field, or if $C_L(B_k)$ is the P_k-field, whereby there is an element $\varphi \neq 0$ in P_k that is simultaneously an even and an odd function of the β_λ, all occurring extensions are algebraic, so $k = n$, where n is the smallest number of square root operations sufficient to compute the coordinates of p. But if $C_L(B_k)$ is the P_k-field, whereby P_k does not contain any element $\neq 0$ that is simultaneously an even and an odd function of the β_λ, one of the extensions induces the transition of a Γ_i-field to the including P_i-field, ("rational" extension), so $k = n + 1$. (General case).

So the theorem of HILBERT must be modified in this sentence: *If B is a basis represented by $z(0,0), e_y(1,0), t_\lambda(\alpha_\lambda, \beta_\lambda), \lambda \in \Lambda$, and $p(\pi_1, \pi_2)$ is a point of $C_Q(B)$, π_1, $\pi_2 \in \Omega(\alpha_\lambda, \beta_\lambda)$, the coordinates with respect to some orthogonal coordinate system, and if n is the smallest number of square root operations sufficient to compute the coordinates π_1, π_2 of p, then p has either 2^n or 2^{n+1} geometrically conjugates over B.*

10. Constructions using circles with fixed radius only.

It is a well known fact that, in congruence geometry, the active operation C_1 producing the circle with given centre and circumference point may be replaced by a weaker one, *producing a circle with given centre and fixed radius* (Cf. HILBERT [1], § 36). It is a not interesting distinction whether we consider that radius as to be borrowed from the set of basis points, or we assume that it must be adjoined to the basis first. In any case only adjunctionless construction fields will be considered; adjunction operations will only be used to procure basis points.

The operation set Q'' will consist of the construction postulates L_1, P_1', L_3, P_4', D^{pp}, D^{ll}, S_1, and the active operation

C_3: *There exists an operation* C_3, *which when applied to a point* a, *produces the circle* $C_3(\dot{a},\widehat{\varrho})$ *with centre* a *and a fixed radius* $\widehat{\varrho}$.

Theorem 22. *When* B *is a basis in the geometry* C *containing two points with a fixed distance* $\widehat{\varrho}$, *the construction field* $C_{Q''}(B)$ *is contained in the construction field* $C_Q(B)$, *and conversely every point and every line of* $C_Q(B)$ *also belongs to* $C_{Q''}(B)$.

Dem. The first statement is obvious. To prove the second we give a construction for the operation P_4' applied to a line $\bar{a}$ and a circle $C_1^{(0)}(\dot{a},b)$ only given by its centre a (a inc $\bar{a}$) and some circumference point b ($b \neq a$). (The remaining operations L_2, D^{par}, and D^{norm} can be derived just as in the construction geometry (C, Q).

P_4': Given two distinct points, a and b, and a line $\bar{a}$, a inc $\bar{a}$, we construct the points of intersection s_1 & s_2 that would be produced applying P_4' to $\bar{a}$ and the circle $C_1^{(0)}(\dot{a},b)$:

If b exc $\bar{a}$: $S_1[P_4'\{L_1(a,b),\ C_3(\dot{a},\widehat{\varrho})\}] = c;$

$$S_1\,[P_4'\,\{\bar{a},\ C_3(\dot{a},\widehat{\varrho})\}] = d_i,\ i = 1,2;$$

$$P_1'\,[L_2\{\text{inc } b,\ /\!/\, L_1(\text{c},d_i)\},\ \bar{a}] = s_1\ \&\ s_2 = P_4'\,\{\bar{a},\ C_1^{(0)}(\dot{a},b)\}.$$

If b inc $\bar{a}$: Let e be any point (given, constructed, or adjoined), e exc $L_1(a,b)$; first we construct the points of intersection t_1 & t_2 of $L_1(\dot{e},a)$ and $C_1^{(0)}(\dot{a},b)$; we select t from t_1 & t_2 and finally we construct the points of intersection s_1 & s_2 of $\bar{a}$ and $C_1^{(0)}(\dot{a},t)$, according to the above construction chain.

Consequently every point and every line of $C_Q(B)$ is constructible in the construction geometry (C, Q'') and belongs to $C_{Q''}(B)$.

CHAPTER VI

DECISION OPERATIONS OF ORDER HILBERT GEOMETRY

1. Objects, relations, operations.

A geometrical system satisfying the axioms on *incidence*, *parallelism*, *congruence*, and *order*, as presented in the first four groups of the HILBERT axiom system (Cf. HILBERT, [1]) is called an *ordered* HILBERT *geometry*; a system satisfying the axioms on incidence, parallelism, and congruence, that contains an ordered HILBERT subgeometry, is called a *partially ordered* HILBERT *geometry*.

A HILBERT geometry will be denoted by H.

It was HILBERT who investigated the set of Euclidean ruler and compasses constructions that are already performable in the present geometry, so independently of the continuity axioms. (Cf. HILBERT, [1], § 36).

Objects: points, lines, circles. Relations: coincidence, incidence, parallelism, congruence, normality (0-relations), and the relation of *betweenness* ($\gtrless$-relation).

We suppose that our geometry can be brought into a one-to-one correspondence with an analytical geometry over a partially ordered field $\varDelta$ with the property: if $\vartheta \in \varDelta$ then also $\sqrt{1+\vartheta^2} \in \varDelta$. The totally real field Ω (Cf. Chapter V, 1, p. 60) is the smallest field contained in the field underlying our geometry.

The operation set Q of Chapter V will be extended to Q^o by adding a decision operation of order, viz. a decision operation concerning the relation of betweenness of three distinct collinear points. Q^o will consist of the operation postulates:

Active operations. L_1 (Chapter II, p. 22), P_1' (Chapter III, p. 28), L_3 (Chapter IV, p. 50), C_1 (Chapter IV, p. 41), P_4' (Chapter V, p. 61).

Decision operations. D^{pp}, D^{ll} (Chapter II, p. 22).

$D^{p \cdot p \cdot p}$: *There exists an operation* $D^{p \cdot p \cdot p}$, *which when applied to three distinct collinear points* a,b,c, *establishes which of them is between the others, so whether we have* $/a,b,c/$, *or* $/b,a,c/$, *or* $/b,c,a/$.

Selection operation: S_1 (Chapter II, p. 22).

Remark. $D^{p\cdot p\cdot p}$ is a decision operation of the second category ($\gtrless$-decision) (Cf. Chapter I, p. 9). With respect to the postulated applicability of $D^{p\cdot p\cdot p}$ and to the applicability of the other $\gtrless$-decision operations we will make a restriction that will be explained in section 3.

Remark. Because congruence geometry can be considered imbedded in HILBERT geometry, and because the operation set Q is contained in Q^o, the construction theorems of Chapter V, 2 hold good in the construction geometry (H, Q^o) too.

2. Some construction theorems.

In this section some $\gtrless$-decision operation theorems will be derived. They are valid in an ordered HILBERT geometry only.

$D^{l\cdot c}$: *It is possible to decide whether a line and a circle intersect or not.*

Dem. In the geometry H as well as in congruence geometry C a line and a circle intersect if and only if the line is incident with the centre of the circle, and consequently the decision operation $D^{l\cdot c}$ is reduced to D^{pl} (0-decision whether a line and a point are incident or excident). In Euclidean geometry we shall meet $D^{l\cdot c}$ as a $\gtrless$-decision operation but here we may consider it a 0-decision operation.

$D^{p\cdot p\cdot l}$: *It is possible to decide whether two distinct points a,b are on the same side or on opposite sides of a line $\bar{a}$, a exc $\bar{a}$, b exc $\bar{a}$.*

Dem. Apply D^{par} to $L_1(a,b)$ and $\bar{a}$; if these lines are parallel, a and b are on the same side of $\bar{a}$, and if not, apply $D^{p\cdot p\cdot p}$ to the triplet $a,b,P_1{}'\,\{L_1(a,b),\bar{a}\}$.

$D^{p\cdot c}$: *It is possible to decide whether a point b is inside or outside a circle $\bar{\bar{a}}$ with centre a, $a \neq b$, b exc $\bar{\bar{a}}$.*

Dem. Apply $D^{p\cdot p\cdot p}$ to the triplet b, $S_1[P_4{}'\,\{L_1(a,b), \bar{\bar{a}}\}] = s_1$ *and* s_2.

In the present ordered geometry similar and opposite orientation can be defined for arbitrary proper triangles, in the usual manner ($\gtrless$-relations). (In reflection geometry and in congruence geometry similar and opposite orientation could be defined for special kinds of triangles; the case of two isosceles right-angled triangles was elaborated especially. The definition in question happened with the aid of a special construction process; in the geometries mentioned similar and opposite orientation of two triangles were 0-relations).

We want to mention the decision operation D^{ort} (decision concerning the orientation of triangles); in reflection geometry and in congruence geometry D^{ort} was a 0-decision operation. Here it will evidently be a $\gtrless$-decision-operation.

D^{ort}: *It is possible to decide whether two proper triangles a,b,c and p,q,r have similar or opposite orientation.*

Dem. We will show that a translation and a rotation are constructible. Let us assume for a moment that a triangle p,b'',c'' has been constructed congruent with the triangle a,b,c and so that b'' and q are on the same side of the point p upon the line $L_1(p,q)$, by a translation (i) and a rotation (ii). Then apply $D^{p.p.l}$ tot the points c'' and r and the line $L_1(p,q)$; if c'' and r are on the same side of $L_1(p,q)$, a,b,c and p,q,r have similar orientation; if c'' and r are on opposite sides of $L_1(p,q)$, the triangles have opposite orientation.

(i) *Translation of a proper triangle a,b,c in such a manner that $a \rightarrow a'$ and a' coincides with a given point p, $p \neq a$.*

The construction does not require any decision operation of the second category; it has been executed in the demonstration of theorem 13, p. 52.

(ii) *Rotation of a triangle p,b',c' around its vertex p in such a manner that the image b'' of b' is incident with a given line $L_1(p,q)$ through p and that b'' is on the same side of p as q.*

$S_2[P_4' \{C_1(p,b'), L_1(p,q)\}] = b_1$ *and* b'' so *that* $|b_1,p,q|$;

$S_2 < P_4' [C_1(p,b'), L_3\{\text{inc}\, p, \perp L_1(b',b'')\}] > = d$ *so that* c' *exc* $L_1(b',d)$;

next we use the active operation P_3 (see p. 43) being performable with the aid of the operations of Q:

$P_3 \{C_1(d,c'), C_1(b'',c')\} = e$; if now e inc $L_1(b'',d)$, then $e = c''$; if e exc $L_1(b'',d)$, then $P_3\{C_1(d,e), C_1(b'',e)\} = c''$; now p,b'',c'' is the image required of the triangle p,b',c'

Remark. Just as in Chapter V the active operation C_1 may be replaced by C_3 (producing circles with a prescribed centre and a fixed radius $\overset{\frown}{\varrho}$); we shall no more deal with this question here.

3. Construction fields.

Consider the adjunctionless construction field $H_{Q^0}(B)$ in an ordered or partially ordered HILBERT geometry H, under the operations of Q^0, upon a basis B. B is assumed to contain two distinct points.

If a construction field is such that the application of the $\gtrless$-decision operations is admitted for all objects, we shall call it an *ordered construction field,* and if not, we shall call it a *partially ordered construction field.* Only certain special cases of partially ordered construction fields will be dealt with.

Definitions. 1. If $\varDelta$ is a field in such a manner that the decision operation $D\gtrless$, establishing whether we have either $\alpha > \beta$ or $\alpha < \beta$ for a couple of distinct elements α, β of $\varDelta$, is applicable to *any* couple of distinct elements of $\varDelta$, $\varDelta$ is called an *ordered field.*

2. If $\varDelta$ is such a field that the decision operation may not be applied to certain couples of distinct elements of $\varDelta$, it is called a *partially ordered field.*

3. If ϑ is an element of a field $\varDelta$, while the decision operation $D\gtrless$ is applicable to ϑ and 0, ϑ is called an *ordered element over* 0.

4. If $\vartheta \neq 0$ is an element of a (partially ordered) field $\varDelta$, while the decision operation $D\gtrless$ is not applicable to ϑ and 0, ϑ is called a *free element over* 0.

Remark: If ϑ is a free element over 0, ϑ is transcendental over the prime field $\varPi$ of the rational numbers.

5. If $\varDelta$ is a field, and ϑ an element of an extension field of $\varDelta$, while the decision operation $D\gtrless$ is not applicable to ϑ and any element δ of $\varDelta$, ϑ is called a *free element over* $\varDelta$. (So ϑ is transcendental over $\varDelta$).

The following kinds of partially ordered construction fields will only be taken into consideration:

Let $z(0,0)$, $e_y(1,0)$, $t_\lambda(\alpha_\lambda,\beta_\lambda)$, $\lambda \in \Lambda$, constitute a basis B; then we shall only consider the cases that all parameters α_λ, β_λ are free elements over 0, or that all β_λ are free elements over 0, whereby the question whether the α_λ are also free elements over 0 or not, is left as it is.

Theorem 23. *When* $z(0,0)$, $e_y(1,0)$, $t_\lambda(\alpha_\lambda,\beta_\lambda)$, $\lambda \in \Lambda$, *constitute a basis* B *of the construction field* $H_{Q}o(B)$ *in the* HILBERT *geometry* H, *a point* $p(\pi_1,\pi_2)$ *belongs to* $H_{Q}o(B)$ *if and only if* π_1, π_2 *are elements of* $\Omega(\alpha_\lambda,\beta_\lambda)$ (HILBERT). (For Ω cf. p. 63).

Dem. *Necessity.*

As in the demonstration of theorem 17, p. 63.

Sufficiency.

The point $e_x(0,1)$ is only constructible uniquely if at least one of the β_λ is an ordered element over 0. If this is not the case we may only speak about *a* point $e_x(0,1)$. The construction of either *a* or *the* point e_x will happen first.

$$L_1(z,e_y) = \bar{y};\ L_3(\text{inc } z,\ \perp \bar{y}) = \bar{x};$$

$$P_4' \{\bar{x},\ C_1(\dot{z},e_y)\} = e_x \,\&\, d_x;$$

if at least one of the β_λ is an ordered element over 0, say $\beta_1 > 0$, then *the* point $e_x(0,1)$ can be constructed uniquely:

$$P_1' \{L_2(\text{inc } t_1,\ /\!/\ \bar{y}),\bar{x}\} = k(0,\beta_1);$$

$$S_2(e_x \,\&\, d_x) = e_x(0,1) \textit{ and } d_x(0,-1) \textit{ so that } /d_x,z,k/ \ (D^{p \cdot p \cdot p}).$$

Now we suppose that *no agreement with respect to the signs of the parameters* β_λ *can be made.* So the β_λ must be either *free elements* over 0, or equal to 0. Then we must use an arbitrary selection to fix a point e_x:

$$S_1\,(e_x \,\&\, d_x) = e_x(0,1).$$

(In this case the points $e_x \,\&\, d_x$ are $H_L(B)$-equivalent, if L is the orthogonal-linear operation set of Chapter IV (L is contained in Q^o)).

The selection of e_x is a *double selection* (Cf. p. 68); the next construction process must be continued in the two possible directions.

Next the operation $\vartheta \rightarrow \sqrt{1+\vartheta^2}$.

Given a point $a(\vartheta,0) \in H_{Q^o}(B)$ we shall construct *the* points $b(\sqrt{1+\vartheta^2},0)$ and $c(0,\sqrt{1+\vartheta^2})$; we suppose $\vartheta \neq 0$.

$$S_2\,[P_4' \{\bar{y},\ C_1(\dot{z},e_y)\}] = d_y(-1,0) \textit{ so that } d_y \neq e_y (D^{pp});$$

$P_1' \{L_3(\text{inc } a,\ \perp \bar{y}),\ L_2(\text{inc } e_x,\ /\!/\ \bar{y})\} = d(\vartheta,1);\ P_4' \{\bar{y},\ C_1(\dot{z},d)\} = b_1 \,\&\, b_2;$

Apply $D^{p \cdot p \cdot p}$ to the triplets d_y,z,b_1 and d_y,z,b_2:

$$S_2(b_1 \,\&\, b_2) = b(\sqrt{1+\vartheta^2},0) \textit{ so that } /d_y,z,b/;$$

$$P_1' \,[L_2\{\text{inc } b,\ /\!/\ L_1(e_y,e_x)\},\ \bar{x}] = c(0,\sqrt{1+\vartheta^2}).$$

So we find that the algebraic operations with respect to the coordinates (and the equation coefficients) of the objects of $H_{Q^o}(B)$ possible in the field $\Omega(\alpha_\lambda,\beta_\lambda)$, are constructible, proving the theorem.

Remark. Without proof we state: $\Omega(\alpha_\lambda,\beta_\lambda)$ is independent under coordinate transformation (similarity transformation) in $H_{Q^o}(B)$; it is called *the field belonging to* $H_{Q^o}(B)$.

4. Automorphisms of construction fields.

Let $z(0,0)$, $e_y(1,0)$, $t_\lambda(\alpha_\lambda,\beta_\lambda)$, $\lambda \in A$, represent a basis B in H; then we have:

Theorem 24. *The adjunctionless construction field $H_Q0(B)$ admits of either one automorphism leaving B pointwise fixed (viz. identity), or two automorphisms (leaving B pointwise fixed) that constitute a two-cyclic group. The latter is the case if and only if all β_λ are either free elements over* 0, *or equal to* 0.

Dem.

1. *None of the β_λ is ordered over* 0. The β_λ are either free elements over 0, or equal to 0. A point $e_x(0,1)$ may be selected arbitrarily. Then the points e_x and $d_x(0,-1)$ are geometrically conjugate over B. The permutations of the points $e_x(0,1)$ and $d_x(0,-1)$ leave the point $b(\sqrt{1+\vartheta^2},0)$ fixed, but they induce permutations of the points $c_1(0,\sqrt{1+\vartheta^2})$ and $c_2(0,-\sqrt{1+\vartheta^2})$; we may say: *the permutations of e_x and d_x induce automorphisms of $H_Q0(B)$ that leave B pointwise fixed.* These automorphisms *constitute a two-cyclic group*; they correspond with the permutations of the isotropic points i_1 & i_2.

The selection of e_x fixes the positive direction of the y-axis. The permutations of e_x & d_x leave B pointwise fixed and consequently also the points $(\varphi_1,0)$ and $(0,\varphi_2)$, where φ_1 is an even and φ_2 is an odd quadratic expression in β_λ with coefficients in $\Omega(\alpha_\lambda)$; but the points $(\psi_1,0)$ and $(-\psi_1,0)$, and the points $(0,\psi_2)$ and $(0,-\psi_2)$ are *interchanged*, where ψ_2 is an even and ψ_1 is an odd quadratic expression in β_λ with coefficients in $\Omega(\alpha_\lambda)$.

Because the β_λ are either free elements over 0, or equal to 0, every element ε of $\Omega(\alpha_\lambda,\beta_\lambda)$ can be uniquely written: $\varepsilon = \varepsilon' + \varepsilon''$, ε' being an even and ε'' being an odd quadratic expression in the β_λ with coefficients in $\Omega(\alpha_\lambda)$. Then we find: *There is a two-cyclic automorphism group of the construction field $H_Q0(B)$; its elements can be represented by the permutations of conjugate points $p_1(\pi_1'+\pi_1'',\pi_2'+\pi_2'')$ and $p_2(\pi_1'-\pi_1'',-\pi_2'+\pi_2'')$, of conjugate lines $\bar{r}_1(y = (\varrho_1'+\varrho_1'')x+\varrho_2'+\varrho_2'')$ and $\bar{r}_2(y = (-\varrho_1'+\varrho_1'')x-\varrho_2'+\varrho_2'')$, of conjugate lines $\bar{s}_1(x=\sigma'+\sigma'')$ and $\bar{s}_2(x = \sigma'-\sigma'')$, and of conjugate circles $\bar{\bar{t}}_1(x^2+y^2+(\tau_1'+\tau_1'')x + (\tau_2'+\tau_2'')y+\tau_3'+\tau_3'' = 0)$ and $\bar{\bar{t}}_2(x^2+y^2+(\tau_1'-\tau_1'')x+(-\tau_2'+\tau_2'')y+\tau_3'-\tau_3'' = 0)$.* (Conjugate = geometrically conjugate over B.)

In the case that all β_λ are equal to 0, these automorphisms represent *identity and reflection with respect to the x-axis.*

2. *At least one of the* β_λ *is an ordered element over* 0, *say* $\beta_1 > 0$. The point $e_x(0,1)$ is constructible uniquely. All objects of $H_{Q^o}(B)$ are constructible uniquely; $H_{Q^o}(B)$ does not admit of any automorphism leaving B pointwise fixed, but identity.

5. Objects constructible uniquely in $H_{Q^o}(B)$.

As a consequence of the demonstration of theorem 24 we find that, if the basis B consisting of the points $z(0,0)$, $e_\lambda(1,0)$, $t_\lambda(\alpha_\lambda,\beta_\lambda)$, $\lambda \in \Lambda$, is such that at least one of the β_λ is an ordered element over 0, any object of $H_{Q^o}(B)$ is constructible uniquely; if all β_λ are either free elements over 0, or equal to 0, an object of $H_{Q^o}(B)$ is constructible uniquely if and only if it is invariant under the automorphisms of $H_{Q^o}(B)$ described above. If all β_λ are equal to 0, the only *points* of $H^u{}_{Q^o}(B)$ (i.e. the subset of $H_{Q^o}(B)$ that is invariant under these automorphisms) are the points of $H_{Q^o}(B)$ incident with the x-axis. If at least one of the β_λ is not equal to 0 we have: a point $p(\pi_1,\pi_2)$ is constructible uniquely in the construction field $H_{Q^o}(B)$ if and only if π_1, π_2 are elements of $\Omega(\alpha_\lambda, \beta_\lambda)$, while π_1 is an even and π_2 is an odd quadratic expression in the β_λ with coefficients in $\Omega(\alpha_\lambda)$.
Remark. Concerning exact construction problems worded with regard to bases of construction fields $H_{Q^o}(B)$ we see again: the problem is solvable if and only if the figure required belongs to $H_{Q^o}(B)$; with a unique description of the figure required corresponds a construction chain with a unique result.

6. Adjunction operations.

The use of the *geometrical adjunction operation* A^g (Cf. p. 12) (e.g., also A^f = adjunction of a *free point* over a basis) will be admitted. It serves to procure the basis points for an adjunctionless construction field $H_{Q^o}(B)$; it will not be needed to construct objects in an adjunctive construction field $H_{Q^o(A)}(B)$ (Cf. p. 14), because it follows from theorem 18 that also in the construction geometry (H, Q^o) a basis containing two distinct points is sufficient: $H_{Q^o(A)}(B) = H_{Q^o}(B)$, so for each B in H we have $H_{Q^o(A)}(B) = H_{Q^o}(B)$.

CHAPTER VII

RULER AND COMPASSES IN EUCLIDEAN GEOMETRY

1. Objects, relations, operations.

A geometrical system satisfying the axioms on *incidence, parallelism, congruence and order*, and, besides, the axioms concerning the existence of the points of intersection of a line and a circle, or of two circles, in the case that certain conditions of order are fulfilled (*circle-axioms*), is called an *ordered Euclidean geometry;* a system satisfying these axioms except the axioms on order, but containing an ordered Euclidean subgeometry, is called a *partially ordered Euclidean geometry.*

A Euclidean geometry will be denoted by E; E may satisfy ARCHIMEDES' axiom but this is not necessary.

Objects and relations as in Chapter VI.

We assume that E can be brought into a one-to-one correspondence with an analytical geometry over a *partially ordered field* Δ with the property: if $\vartheta \in \Delta$ and $\vartheta > 0$, then $\sqrt{\vartheta} \in \Delta$. The field Σ whose elements are derived from those of the field Π of the rational numbers by a finite number of additions, subtractions, multiplications, divisions, and of operations $\vartheta \rightarrow \sqrt{\vartheta}\,(\vartheta > 0)$, is the smallest field contained in the field underlying our geometry E.

We introduce the set V of operations postulated:

Active operations.

L_1 (Chapter II, p. 22); P_1' (Chapter III, p. 28); C_1 (Chapter IV, p. 41); (Univalent operations);

P_4: *There exists an operation* P_4, *which when applied to a line* $\bar{a}$ *and a circle* $\bar{\bar{a}}$, *having at least one point in common, produces both (possibly coinciding) points of intersection* $P_4(\bar{a},\bar{\bar{a}}) = P_4(\bar{\bar{a}},\bar{a})$ *simultaneously.* (Bivalent operation.)

P_5: *There exists an operation* P_5, *which when applied to two circles* $\bar{\bar{a}}$ *and* $\bar{\bar{b}}$, *having at least one point in common, produces both* *(possibly coinciding) points of intersection* $P_5(\bar{a},\bar{\bar{b}})$ *simultaneously.* (Bivalent operation).

(We shall see that P_5 is dependent on P_4(Cf., e.g., the PONCELET-STEINER theorem); for the sake of convenience we shall use P_5 as a postulated operation.)

Decision operations. D^{pp}, D^{ll} (Chapter II, p. 22); $D^{p.p.p}$(Chapter VI, p. 84). $D^{p.p.p}$ will only be valid in ordered geometries.
Selection operation: S_1 (Chapter II, p. 22).

2. Construction theorems.

It will not be necessary to describe explicitly the usual contructions by ruler and compasses for the active operations L_2 (parallels) and L_3 (normals); suffice it to say that all operations possible in the construction geometry (E, Q^0) can be performed in the present construction geometry (E,V), where Q^0 is the operation set introduced in HILBERT geometry (Chapter VII). So the following operations are possible: P_4' (p. 61), D^{pl}(*p.* 23), D^{pc}, D^{cc} (p. 42), L_2, D^{par} (p. 28/29), L_3, D^{norm}(p. 42), C_2(p. 43), P_2(p. 42), P_3(p. 43), $D^{p.p.l}$, $D^{p.c}$, D^{ort} (pp. 85, 86).

Furthermore we have:

$D^{c.c}$: *It is possible to decide whether two distinct circles intersect, or touch, or have no point in common.*

Dem. Given two circles $\bar{\bar{a}} = C_1(\dot{a},p)$ and $\bar{\bar{b}} = C_1(\dot{b},q)$, $a \neq b$, then S_1 $[P_4'\{L_1(a,b), \bar{\bar{a}}\}] = s_1$ *and* s_2; S_1 $[P_4'\{L_1(a,b), \bar{\bar{b}}\}] = t_1$ *and* t_2; apply D^{pp} to the four point couples s_1,t_1; s_2,t_1; s_1,t_2; s_2,t_2; apply $D^{p.p.p}$ to the triplets s_1,s_2,t_1 and s_1, s_2, t_2.

$D^{l.c}$: *It is possible to decide whether a line and a circle intersect, or touch, or have no point in common.*

Dem. Given a line $\bar{a}$ and a circle $\bar{\bar{a}} = C_1(\dot{a},b)$, we apply D^{pl} to a and $\bar{a}$; if a inc $\bar{a}$, then $\bar{a}$ and $\bar{\bar{a}}$ intersect; if a exc $\bar{a}$, then $P_1'\{L_3(\text{inc } a, \perp \bar{a}), \bar{a}\} = s$; $S_1[P_4'$ $\{L_3(\text{inc } a, \perp \bar{a}), \bar{\bar{a}}\}] = t_1$ *and* t_2; apply D^{pp} to the point couples s,t_1 and s,t_2; if these are proper couples, then apply $D^{p.p.p}$ to the triplet s,t_1,t_2.

Remarks. We may use P_4' without preceding decision operation, because P_4' is a *non-conditional operation.*

The decision operations $D^{p.p.p}$, $D^{p.p.l}$, $D^{p.c}$, D^{ort}, $D^{c.c}$, $D^{l.c}$ are $\gtrless$-decision operations; the others are 0-decision operations. The $\gtrless$-decision operations are only available in ordered geometries.

3. Construction fields.

Consider the adjunctionless construction field $E_V(B)$ in the geometry E under the operations of V upon a basis B. Consider a basis containing *at least two distinct points*. Select the points $z(0,0)$, $e_y(1,0)$ from B; the remaining basis points get parametrical coordinates $t_\lambda(\alpha_\lambda,\beta_\lambda)$, $\lambda \in \Lambda$, Λ some index system. (Coordinates with respect to an orthogonal Cartesian coordinate system).

We remind the reader that in the case of a basis consisting of a set of (at least two distinct) collinear points the orthogonal-linear construction field will be the *adjunctive* construction field $E_{L(A)}(B)$ (Cf. p. 49); nevertheless we shall generally use the notation $E_L(B)$.

Let $\Sigma(\alpha_\lambda,\beta_\lambda)$ be the field the elements of which are derived from those of the field $\Pi(\alpha_\lambda,\beta_\lambda)$ by a finite number of additions, subtractions, multiplications, divisions, and operations $\vartheta \rightarrow \sqrt{\vartheta}$ $(\vartheta > 0)$.

Concerning the $\gtrless$-decision operations we will again make the restriction that their validity will not always be admitted. We will consider *ordered* and some kinds of *partially ordered* construction fields. (Cf. p. 87).

Theorem 25. *When* $z(0,0)$, $e_y(1,0)$, $t_\lambda(\alpha_\lambda,\beta_\lambda)$, $\lambda \in \Lambda$, *constitute a basis* B *of a construction field* $E_V(B)$ *a point* $p(\pi_1,\pi_2)$ *belongs to* $E_V(B)$ *if and only if* π_1, π_2 *are elements of* $\Sigma(\alpha_\lambda,\beta_\lambda)$.

Dem. *Necessity.*

The basis points satisfy the condition of the theorem. The operations P_4 and P_5 are reflected either by rational operations or by quadratic operations (extraction of a square root), and the remaining operations are reflected by rational operations with respect to the coordinates of points and to the equation coefficients of lines and circles.

Sufficiency.

The point $e_x(0,1)$ is only constructible (uniquely) if at least one of the β_λ is an ordered element over 0. If this is not the case, we may only speak about *a* point (0,1).

$$L_1(z,e_y) = \bar{y};\ L_3(\text{inc } z,\ \perp \bar{y}) = \bar{x};$$
$$P_4'\,\{\bar{x},\ C_1(\dot{z},e_y)\} = e_x \,\&\, d_x;$$

if at least one of the β_λ is an ordered element over 0, say $\beta_1 > 0$, then *the* point $e_x(0,1)$ can be selected uniquely:

$$P_1'\,\{L_2(\text{inc } t_1,\ /\!/\ \bar{y}),\ \bar{x}\} = k(0,\beta_1);$$
$$S_2\,(e_x \,\&\, d_x) = e_x(0,1) \text{ so that } |d_x,z,k|\ (D^{\mathfrak{p}\cdot\mathfrak{p}\cdot\mathfrak{p}}).$$

Now we suppose that no agreement with respect to the signs of the parameters β_λ can be made. So the β_λ must be either free elements over 0, or equal to 0. We use an arbitrary selection to fix a point e_x:

$$S_1\,(e_x \mathbin{\&} d_x) = e_x(0,1).$$

(In this case the points e_x & d_x are $E_L(B)$-equivalent, if L is the orthogonal-linear operation set introduced in Chapter IV; L is contained in V).

In the case that e_x has been constructed uniquely as well as in the case that e_x has been fixed by selection, the following construction process always is possible.

Given a point $a(\vartheta,0) \in E_V(B)$, $\vartheta > 0$, we construct the points $b(\sqrt{\vartheta},0)$ and $c(0,\sqrt{\vartheta})$. (We suppose $\vartheta \neq 1$).

We have $/d_y(-1,0),\ z,a/$; $S_1[P_5\,\{C_1(\dot{d}_y,a),\ C_1(\dot{a},d_y)\}] = s_1$ *and* s_2;

$$P_1\{L_1(s_1,s_2),\ \bar{y}\} = m(\tfrac{1}{2}(\vartheta-1),0);\ S_1[P_4\{\bar{x},C_1(\dot{m},a)\}] = t(0,\pm\sqrt{\vartheta});$$

$$S_2\,[P_4\{C_1(\dot{z},t),\ \bar{y}\}] = b \textit{ so that } /d_y,z,b/.$$

In the case that e_x has been found by *selection*, the following part of the construction must be performed in the two possible directions, taking e_x and d_x for the point (0,1), because the selection of e_x must be considered a *double* one; then we find a set of two geometrically conjugate points over B, one of which may be chosen to be the point $c(0,\sqrt{\vartheta})$:

$$S_1 < P_1'\,[L_2\{\text{inc}\ b,\ //\,L_1(e_y,e_x)\},\ \bar{x}] > \; = c(0,\sqrt{\vartheta}) \textit{ and } c'(0,-\sqrt{\vartheta}).$$

So we find that the algebraic operations possible in the field Σ $(\alpha_\lambda,\beta_\lambda)$ are constructible with respect to the coordinates of points (and to the equation coefficients of lines and circles) in the construction field $E_V(B)$, proving the theorem.

Remark. Without proof we state: $\Sigma(\alpha_\lambda,\beta_\lambda;\ \lambda \in \Lambda)$ is independent under coordinate transformation (similarity transformation) in $E_V(B)$; it is called *the field belonging to* $E_V(B)$.

Again consider a basis B consisting of the points $z(0,0)$, $e_y(1,0)$, $t_\lambda(\alpha_\lambda,\beta_\lambda)$, $\lambda \in \Lambda$. If the parameters $\alpha_\lambda,\beta_\lambda$ are free elements over Π, Π being the prime field of the rational numbers, no inequality $\alpha_\lambda \gtrless \delta$ or $\beta_\lambda \gtrless \delta$ is decidable for any δ in Π; the operation $\vartheta \to \sqrt{\vartheta}$ can only be applied to elements ϑ of $\Sigma(\alpha_\lambda,\beta_\lambda)$ that are definitely non-negative functions of the α_λ and β_λ. It is a well known fact that any definitely non-negative element in an ordered real field can be written as the sum of squares of field elements. (Cf. v. D. WAERDEN [1], I, § 71). This means: $\Sigma(\alpha_\lambda,\beta_\lambda) = \Omega(\alpha_\lambda,\beta_\lambda)$.

In this case all objects of the construction field $E_V(B)$ belong already to $E_{Q^o}(B)$, where Q^o is the operation set introduced in HILBERT geometry, Chapter VI. The only branching operation of Q^o is P_4', producing the points of intersection of a circle and a line passing through the centre of the circle. P_4' is independent of decisions of order.

4. Automorphisms of construction fields. Objects constructible uniquely in E_V (B).

Let $z(0,0)$, $e_y(1,0)$, $t_\lambda(\alpha_\lambda,\beta_\lambda)$, $\lambda \in A$, represent a basis B in E; the following theorem holds, both theorem and proof corresponding with theorem 24 in HILBERT geometry.

Theorem 26. *The adjunctionless construction field $E_V(B)$ admits of either one automorphism leaving B pointwise fixed (viz. identity), or two automorphisms (leaving B pointwise fixed) that constitute a two-cyclic group. The latter is the case if and only if all β_λ are either free elements over* 0, *or equal to* 0.

Dem.

1. *None of the β_λ is ordered over* 0. The β_λ are either free elements over 0 or equal to 0. A point $e_x(0,1)$ may be selected arbitrarily; then the points e_x and $d_x(0,-1)$ are geometrically conjugate over B.

The permutations of the points e_x and d_x induce automorphisms of $E_V(B)$ that leave B pointwise fixed. These automorphisms constitute a two-cyclic group, corresponding with the permutation group of the isotropic points i_1 & i_2.

Every element ε of $\Sigma(\alpha_\lambda,\beta_\lambda)$ can be uniquely written $\varepsilon = \varepsilon'+\varepsilon''$, ε' being an even and ε'' being an odd quadratic expression in the β_λ with coefficients in $\Sigma(\alpha_\lambda)$. *The automorphisms of $E_V(B)$ can be represented by the permutations of conjugate points $p_1(\pi_1'+\pi_1'', \pi_2'+\pi_2'')$ and $p_2(\pi_1'-\pi_1'', -\pi_2'+\pi_2'')$, of conjugate lines $\bar{r}_1(y = (\varrho_1'+\varrho_1'')\ x+\varrho_2'+\varrho_2'')$* and *$\bar{r}_2(y = (-\varrho_1'+\varrho_1'')\ x-\varrho_2'+\varrho_2'')$, of conjugate lines $\bar{s}_1(x = \sigma' + \sigma'')$ and $\bar{s}_2(x = \sigma'-\sigma'')$, and of conjugate circles $\bar{\bar{t}}_1(x^2+y^2+(\tau_1'+\tau_1'')\ x+(\tau_2'+\tau_2'')\ y + \tau_3'+\tau_3'' = 0)$ and $\bar{\bar{t}}_2(x^2+y^2 + (\tau_1'-\tau_1'')\ x + (-\tau_2'+\tau_2'')\ y+\tau_3'-\tau_3'' = 0)$.*

(Conjugate = geometrically conjugate over B.)

In the case that all β_λ are equal to 0, these automorphisms represent *identity and reflection with respect to the x-axis.*

2. *At least one of the β_λ is an ordered element over* 0, say $\beta_1 > 0$.

The point $e_x(0,1)$ is constructible uniquely. All objects of $E_V(B)$ are constructible uniquely; $E_V(B)$ does not admit of any automorphism leaving B pointwise fixed, but identity.

As a consequence of the demonstration of theorem 26 we find that, if at least one of the β_λ is an ordered element over 0, any object of $E_V(B)$ is constructible uniquely; if all β_λ are either free elements over 0, or equal to 0, an object of $E_V(B)$ is constructible uniquely if and only if it is invariant under the automorphisms of $E_V(B)$ mentioned above. Let the set of uniquely constructible objects of $E_V(B)$ be represented by $E^u{}_V(B)$; then a point $p(\pi_1,\pi_2)$ belongs to $E^u{}_V(B)$ if and only if π_1,π_2 are elements of the field $\Sigma(\alpha_\lambda, \beta_\lambda)$, while π_1 is an even and π_2 is an odd quadratic expression in the β_λ with coefficients in $\Sigma(\alpha_\lambda)$.

Remark. Concerning exact construction problems worded with respect to a basis of a construction field $E_V(B)$ we see again: the problem is solvable if and only if the required figure belongs to $E_V(B)$; with a unique description of the figure required corresponds a construction chain with a unique result.

5. Adjunction operations.

The use of the *geometrical adjunction operations* A^g(Cf. p. 12) (e.g., also A^f = adjunction of a *free* point over a basis) will be admitted. They serve to procure the basis points for an adjunctionless construction field $E_V(B)$; they will not be needed to construct objects in an adjunctive construction field $E_{V(A)}(B)$ (Cf. pp. 14, 15).

Indeed, by a little modification of the proof of theorem 18, that was worded with regard to fields of the type Ω, we may prove that given a basis containing two distinct points, no adjunction operations are needed to construct objects of $E_{V(A)}(B)$, so that such a basis is *sufficient*: $E_V(B) = E_{V(A)}(B)$; so for any B in E we have: $E_{V(A)}(B) = E_V(B)$.

6. The PONCELET-STEINER theorem.

The well known PONCELET-STEINER theorem states that the use of only *one circle* suffices in order to construct any point or any line of the construction field $E_V(B)$. This means that the active operation P_5 (points of intersection of two circles) is superfluous. In its rigid form this statement is not quite correct: e.g., when the basis B consists of *two distinct points*, B is sufficient in (E,V), but if we cancel P_5 from V this will no longer be the case. Consequently we should restrict ourselves to bases that contain three non-collinear points; the adjunctionless operation set V' can be introduced, consisting of the postulated operations L_1, P_1', C_1, P_4, D^{pp}, D^{ll}, $D^{p \cdot p \cdot p}$, S_1. Condition: C_1 may be used only *once*.

We shall not go into the Euclidean construction processes whereby only one circle is used; we may refer to various books on construction geometries, e.g., BIEBERBACH [1], § 5.

7. Construction geometry of normals. (ii)

This section stands in close connection with Chapter IV, 5, p. 49. In Euclidean geometry the construction geometry (E,N^o) will be considered built with the means of construction available when using the constructive instruments *ruler* and *right square*. The latter admits of two different kinds of use:

(*i*) Given a point a and a line $\bar{a}$ the first use of the instrument produces the line passing through a normal to $\bar{a}$ (Active operation L_3).

(*ii*) *Given two distinct points a and b, and a line $\bar{a}$, the second use of the instrument produces the point(s) $p_{1,2}$ of $\bar{a}$ so that $a\hat{p}_ib = 90^\circ$*

p_1 *&* p_2 are the points of intersection of a circle with a and b as *diametrical points* and the line $\bar{a}$. With the aid of the first using of the right square the midpoint m of the point couple a,b is constructible; conversely, when a circle is determined by centre and circumference point, we may construct the opposite diametrical point of the latter, using a reflection with respect to the centre, similarly performable in the construction geometry (E,N), N being the operation set used in Chapter IV (Reflection geometry), section 5. Consequently the second using of the right square corresponds with the active operation P_4; its performability depends on the possibility of deciding whether a line and a circle have a point in common or not; so the decision operation $D^{l \cdot c}$ must be postulated. (We remind the reader that circles will not occur as objects in the present construction geometry.)

The set N^0 of construction operations postulated in E will consist of:

Active operations: L_1 (Chapter II, p. 22); P_1' (Chapter III, p. 28); L_3 (Chapter IV, p. 50); P_4 (Chapter VII, p. 91);

Decision operations: D^{pp}, D^{ll} (Chapter II, p. 22);

$D^{l.c}$: *There exists an operation* $D^{l.c}$, *which when applied to a line* $\bar{a}$ *and a circle* $\bar{a}^{(0)}$ *(determined by two diametrical points), establishes whether* $\bar{a}$ *and* $\bar{a}^{(0)}$ *have any point in common or not.*

Selection operation: S_1 (Chapter II, p. 22).

N^0 may be extended to $N^0(A)$ by adding the ordinary adjunction operations A^g.

Decision theorem:

$D^{p.p.p}$: *It is possible to decide which of three distinct collinear points is between the others.*

Dem. Given three distinct (D^{pp}) collinear (D^{pl}) points a_1, a_2, a_3, apply $D^{l.c}$ to $L_3\{\text{inc } a_i, \perp L_1(a_1,a_2)\}$ and to the circle with the points a_j and a_k as diametrical points ($i, j, k = 1, 2, 3, i \neq j \neq k \neq i$).

As has been mentioned in section 6 the active operation P_5 is dependent on the remaining operations of the set V; consequently it has been proved that the operations of V are performable in the construction geometry (E,N^0), except C_1; conversely the operations of N_0 are performable in (E,V). It follows:

Theorem 27. *When B is a basis in E containing three non-collinear points, the construction fields* $E_V(B)$ *and* $E_{N^0}(B)$ *are identical, apart from the objects: circles.*

THE MOHR-MASCHERONI CONSTRUCTION GEOMETRY IN THE EUCLIDEAN PLANE

I. Objects, relations, operations.

In the MOHR-MASCHERONI construction geometry a pair of compasses is the only instrument used. The occurring objects are *points* and *circles*. A line is determined by two distinct points; lines will be denoted by $L_1^{(o)}, L_2^{(o)}$,

A Euclidean geometry will generally be denoted by E; if it satisfies ARCHIMEDES' axiom especially, it will be denoted by E', and if we want to stress that E is a non-Archimedean geometry we will write E''

The well known MOHR-MASCHERONI theorem states that any linear and quadratic construction (with the exception of the construction of *lines*) possible in the construction geometry (E,V) can be performed with the aid of compasses only; we shall prove that this theorem holds if and only if the geometry in question is *Archimedean*. Furthermore we shall prove that in certain non-Archimedean geometries an analogous theorem may hold on the condition that we admit the successive adjunctions of an enumerable number of determinate points.

We assume that E can be brought into a one-to-one correspondence with an analytical geometry over an *ordered field* Δ (*Archimedeanly ordered* for E') with the property: if $\vartheta \in \Delta$ and $\vartheta > 0$, then also $\sqrt{\vartheta} \in \Delta$.

The adjunctionless operation set M will consist of the following postulated operations.

Active operations: C_1 (Chapter IV, p. 41) (Univalent operation); P_5 (Chapter VII, p. 91) (Bivalent operation).

Decision operations: D^{pp} (Chapter II, p. 22);

$D^{c.c}$: *There exists an operation* $D^{c.c}$, *which when applied to two* *distinct circles $\bar{a}$ and $\bar{\bar{b}}$, establishes whether $\bar{a}$ and $\bar{\bar{b}}$ have at least one point in common or not.*

Selection operation: S_1 (Chapter II, p. 22).

It will be proved later on that *only determinate adjunctions* (A^d) are allowed in order to form bases for construction fields, so that the Mohr-Mascheroni theorem holds. This means that only *ordered* construction fields are to be considered.

2. The MOHR-MASCHERONI construction theorem.

Theorem 28. *Let B be a basis consisting of at least two distinct points, in an Archimedean geometry E', in such a manner that the Euclidean construction field $E'_V(B)$ under the operations of V (Chapter VII) is an ordered construction field; then the adjunctionless construction fields $E'_M(B)$ and $E'_V(B)$ are identical, apart from the objects: lines.*

Dem. (1): The operations of M are performable in the construction geometry (E',V).

(2) We shall derive the possibility of the operations of V from the operations of M. The coherence of these derivations is explained by the diagram of figure 10.

P$_3$: Given a point a and two circles $C_1(\dot{b},a)$ and $C_1(c,a)$, $b \neq c$; construction of the "other" point of intersection d_1 of these circles: $S_2\,[P_5\{C_1(\dot{b},a),\, C_1(\dot{c},a)\}] = d_1$ *and* d_2 *so that* $d_2 = a$ (D^{pp}).

This operation may be denoted by $Refl_{L_1(o)(b,c)}(a) = d_1$ (*reflection of a point with respect to a line*).

P$_2$': Given two distinct points a and b; construction of the "other" point of intersection e of the circle $C_1(\dot{b},a)$ and the line $L_1^{(o)}(a,b)$, where the line passes through the centre of the circle:

$S_1\,[P_5\{C_1(\dot{a},b),\, C_1(\dot{b},a)\}] = c$; $Refl_{L_1(o)(b,c)}(a) = d$; $Refl_{L_1(o)(b,d)}(c) = e$.

This operation may be denoted by $Refl_b(a) = e$ (*reflection of a point with respect to another point*).

D^{pl}: Given three distinct points a,b,c; decision whether a is incident or excident with $L_1^{(o)}(b,c)$: apply D^{pp} to a and the point $Refl_{L_1(o)(b,c)}(a) = d_1$.

L$_2^{(o)}$: Given three non-collinear (D^{pl}) points a,b,c; construction of the line through a parallel to $L_1^{(o)}(b,c)$:

$Refl_b(a) = d$; $Refl_c(d) = e$; $L_1^{(o)}(a,e)$ is the parallel required.

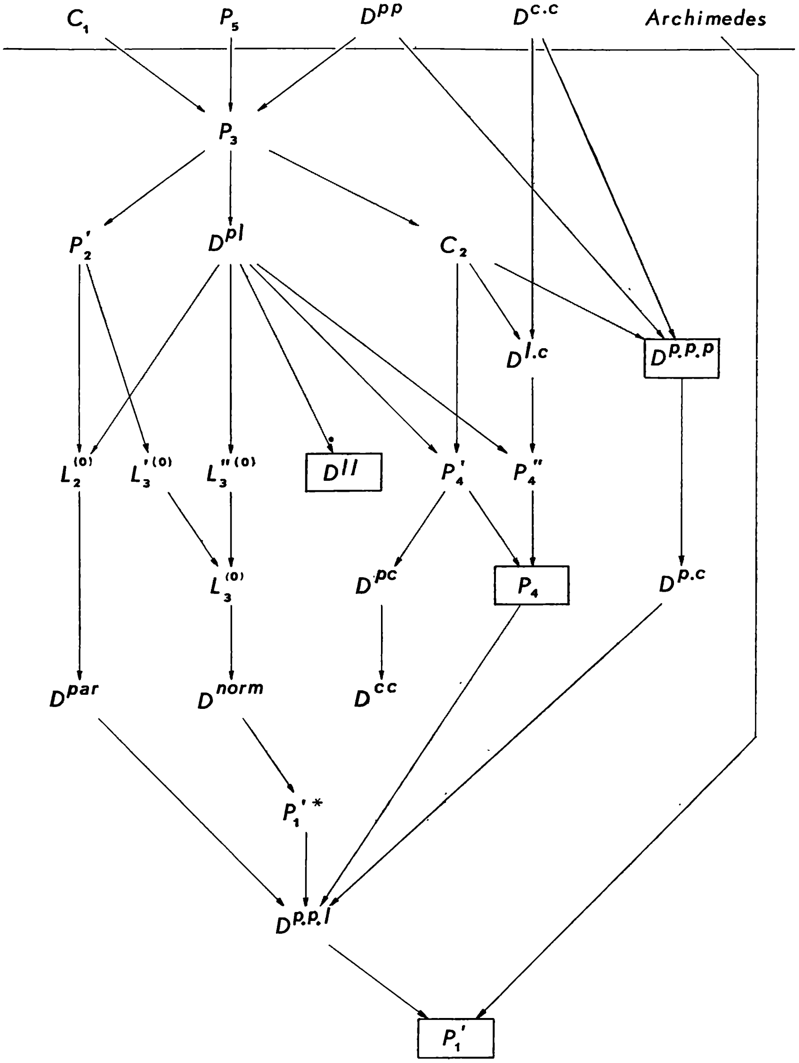

FIG. 10

Coherence of the derivation of the operations of V from the postulated operations of M.

D^{par}: Since the construction of a parallel is possible, it is also posible to decide whether two distinct (D^{pl}) lines are parallel or not (the lines are given by means of point couples).

L$_3'^{(o)}$: Let a,b be two distinct points; construction of the line passing through a normal to $L_1^{(o)}(a,b)$:
$Refl_a(b) = c$; $S_1[P_5\{C_1(\dot{c},b), C_1(\dot{b},c)\}] = d_1$ *and* d_2; then $L_1^{(o)}(d_1,d_2)$ is the normal required.

L$_3''^{(o)}$: Given a point a and a line $L_1^{(o)}(b,c)$ excident with a; construction of the line passing through a normal to $L_1^{(o)}(b,c)$:
$Refl_{L_1(o)(b,c)}(a) = d$; then $L_1^{(o)}(a,d)$ is the normal required.

L$_3^{(o)}$: Given three points a,b,c, $b \neq c$, it is possible to construct the line through a normal to $L_1^{(o)}(b,c)$ by virtue of the performability of $L_3'^{(o)}$ and $L_3''^{(o)}$, being complementary.

D^{norm}: Since the construction of a normal is possible, it is also possible to decide whether two lines are normal or not.

P$_1'$*****: *Given two normal lines* $L_1^{(o)}(a,b)$ *and* $L_1^{(o)}(c;d)$ it is possible to construct their point of intersection s.

Dem. $Refl_{L_1(o)(c,d)}(b) = b_1$; if $b_1 = b$, then b is already the point of intersection required; if not, then $Refl_b(b_1) = b_2$;

$$S_1[P_5\{C_1(\dot{b}_1,b), C_1(\dot{b}_2,b_1)\}] = e_1 \textit{ and } e_2;$$

$Refl_{L_1(o)_{(e_1,e_2)}}(b_1) = s$, this being the point required.

D^{ll}: Given two lines $L_1^{(o)}(a,b)$ and $L_1^{(o)}(c,d)$; decision whether they coincide or not: apply D^{pl} to $L_1^{(o)}(a,b)$ and to c, as well as to d.

C$_2$: Given three distinct points a,b,c; construction of the circle $C_2(\dot{a},\widehat{bc})$ with centre a and radius $\widehat{bc}$:
$S_1\ [P_5\ \{C_1(\dot{a},b), C_1(b,a)\}] = k_1$ *and* k_2; apply D^{pp} to c and k_i, $i = 1,2$. If c $= k_1$ or $c = k_2$, then $C_1(\dot{a},c)$ is the circle required; if not, then $Refl_{L_1(O)_{(k_1,k_2)}}(c) = d$; $C_1(\dot{a},d)$ is the circle required.

P$_4'$: Given a point a and a circle $\bar{\bar{b}}$ with centre b, $a \neq b$; construction of the points of intersection of $\bar{\bar{b}}$ and the line $L_1^{(o)}(a,b)$: let p be any point of the circumference of $\bar{\bar{b}}$; if p inc $L_1^{(o)}(a,b)$, then p is already one of the points required and the other one is found by simple reflection: $Refl_b(p) = p'$ If p exc $L_1^{(o)}(a,b)$:
$Refl_{L_1(o)(a,b)}(p) = q$;
$S_2\ [P_5\ \{C_2(\dot{b},\widehat{pq}), C_1(\dot{p},b)\}] = c_1$ *and* c_2 *so that* $c_1 = b$:

$S_2\,[P_5\,\{C_2(\dot{b},\widehat{pq}),\ C_1(\dot{q},b)\}] = d_1$ *and* d_2 *so that* $d_1 = b$;
$S_1\,[P_5\,\{C_1(\dot{c}_2,q),\ C_1(\dot{d}_2,p)\}] = e$;
$P_5\,\{C_2(\dot{c}_2,\widehat{be}),\ C_2(\dot{d}_2,\widehat{be})\} = s_1\ \&\ s_2$, being the points required,

So the midpoints of the arcs of a circle between two circumference points are found simultaneously.

$\mathbf{D}^{pc}$: Given a point a and a circle $\bar{\bar{b}}$ with centre b, $a \neq b$; decision whether a inc $\bar{\bar{b}}$ or a exc $\bar{\bar{b}}$: apply D^{pp} to a and to the points $P_4'\{L_1^{(o)}(a,b),\ \bar{\bar{b}}\}$.

$\mathbf{D}^{cc}$: Given two circles $\bar{\bar{a}}$, $\bar{\bar{b}}$ with respective centres a,b; decision whether they coincide or not: apply D^{pp} to a and b, and D^{pc} to $\bar{\bar{a}}$ and to some circumference point of $\bar{\bar{b}}$.

$\mathbf{D}^{l.c}$: Given a circle $C_1(\dot{a},b)$ and a line $L_1^{(o)}(c,d)$; decision whether circle and line have a point in common or not: if a inc $L_1^{(o)}(c,d)$, they have a point in common; if a exc $L_1^{(o)}(c,d)$, then $Refl_{L_1(o)(c,d)}(a) = e$, and apply $D^{c.c}$ to $C_1(\dot{a},b)$ and $C_2(\dot{e},\widehat{ab})$.

$\mathbf{P}_4''$: Given a circle $C_1(\dot{a},b)$ and a line $L_1^{(o)}(c,d)$, a exc $L_1^{(o)}(c,d)$; construction of the points of intersection of circle and line, if they exist $(D^{l.c})$:

$Refl_{L_1(o)(c,d)}(a) = e$; $P_5\{C_1(\dot{a},b),\ C_2(\dot{e},\widehat{ab})\} = s_1\ \&\ s_2$, being the points of intersection required.

$\mathbf{P}_4$: Given two points a,b and a circle $\bar{\bar{a}}$; *it is possible to construct the points of intersection of* $\bar{\bar{a}}$ *and* $L_1^{(o)}(a,b)$, *if they exist,* by virtue of the performability of P_4' and P_4'', being complementary.

$\mathbf{D}^{p.p.p}$: Given three distinct collinear points a, b, c; decision which of them is between the others:

Apply $D^{c.c}$ to $C_1(\dot{a},b)$ and $C_2(\dot{c},\widehat{ab})$ and to the five other pairs of circles arising from these by interchanging the points a,b,c. If $C_1(\dot{a},b)$ and $C_2(\dot{c},\widehat{ab})$ touch, then $/a,b,c/$; if they have no point in common, then either $/a,b,c/$ or $/b,a,c/$; suppose that we have the latter case, then $Refl_c(a) = a_1$; if now $C_1(\dot{b},c)$ and $C_2(\dot{a}_1,\widehat{bc})$ do not intersect either, then $/a,b,c/$, but if they do, then $/b,a,c/$.

$\mathbf{D}^{p.c}$: Given a point a and a circle $\bar{\bar{b}}$ with centre b; decision whether a is inside or outside $\bar{\bar{b}}$: Apply D^{pp} and/or $D^{p.p.p}$ to a and the points $P_4'\{L_1^{(o)}(a,b),\ \bar{\bar{b}}\}] = c_1\ \&\ c_2$.

$\mathbf{D}^{p.p.l}$: Given two distinct points a,b and a line $L_1^{(o)}(c,d)$ excident with both a and b; decision whether a and b are on the same side or on opposite sides of $L_1^{(o)}(c,d)$:

Apply D^{par} to $L_1^{(0)}(a,b)$ and $L_1^{(0)}(c,d)$; then, if these lines are not parallel, $P_1'^*[L_3^{(0)}\{\text{inc } a, \perp L_1^{(0)}(c,d)\}, L_1^{(0)}(c,d)\} = c_1$;
$P_1'^*[L_3^{(0)}\{\text{inc } b, \perp L_1^{(0)}(c,d)\}, L_1^{(0)}(c,d)] = d_1$;
suppose $\widehat{bd_1} > \widehat{ac_1}$ $(C_2, P_4, D^{p \cdot c})$;
$Refl_{L_1(0)(c,d)}(b) = b_1$; apply $D^{p \cdot c}$ to b_1 and $C_1(\dot{a},b)$.

Finally we shall derive P_1', using ARCHIMEDES' axiom. To that end we first introduce the concept of **Archimedean construction chain:**

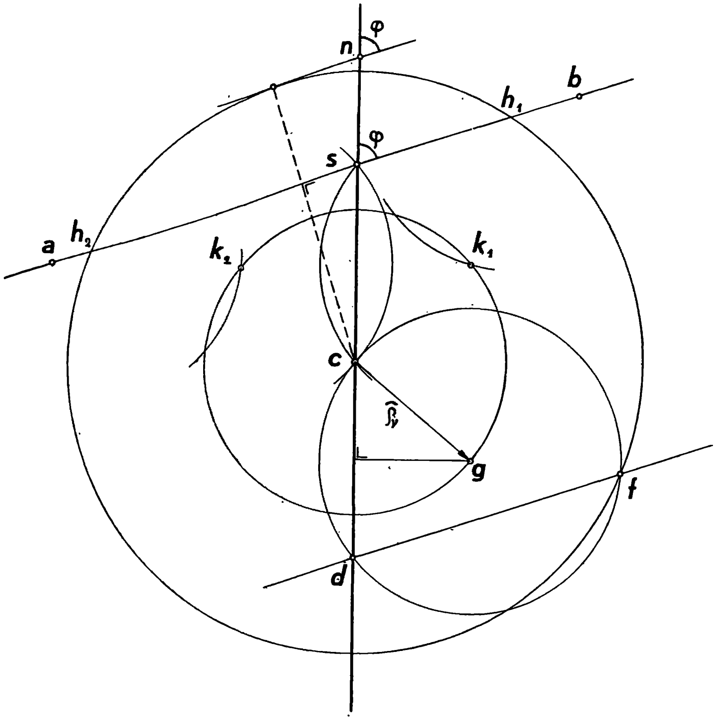

FIG. 11

If $\hat{\varphi}$ be the (unknown) angle between $L_1^{(o)}$ (a, b) and $L_1^{(o)}$ (c, d), then $\widehat{cf} = 2\widehat{\rho_\nu} \sin \hat{\varphi}$, so $\widehat{cn} = 2\, \widehat{\rho_\nu}$.

Definition. Let a_0,a_1 be two distinct points of E. *The Archimedean construction chain upon the ordered point couple a_0,a_1 consists of an infinitely continued series of alternating operations*

$$C_1(\dot{a}_0,a_\nu) = C_1(\dot{a}_0, \widehat{\varrho_\nu}); \; Refl_{a_\nu}(a_{\nu\text{-}1}) = a_{\nu+1}; \; \nu = 1,2,3,\ldots$$

So we find a series of circles with centre a_0 and with radii $\widehat{\varrho_\nu} = \nu\widehat{a_0 a_1}$. (The notion of Archimedean construction chain is theoretical, not constructive; at most a finite number of steps of an Archimedean construction chain is performable constructively.)

Corollary. If E' is an Archimedean geometry, a_0, a_1 a point couple in E', any point p of E' is inside at least one of the circles of the Archimedean construction chain upon a_0, a_1 (p is *Archimedean* over a_0, a_1).

$\mathbf{P}_1'$: Given two distinct non-parallel lines $L_1^{(o)}(a,b)$ and $L_1^{(o)}(c,d)$ in the Archimedean geometry E', then we construct the point of intersection of these lines.

We shall give the construction chain corresponding with the construction method communicated by H. LEBESGUE [1], p. 27. (Cf. figure 11).

The points a,c,d are assumed to be non-collinear (otherwise a would be the point of intersection).

$\mathit{Refl}_a(d) = d_1$; $\mathit{Refl}_b(d_1) = e$, then $L_1^{(o)}(d,e)$ is the line through d parallel with $L_1^{(o)}(a,b)$.

Let the unknown point of intersection be called s; now apply a number of operations of the Archimedean construction chain to the ordered point couple $c = c_o$, $d = c_1$.

According to ARCHIMEDES' axiom, so large a radius $\widehat{\varrho}_\nu$ will be found in this construction chain that the unknown point s is inside the circle with centre c_o and with radius $2\widehat{\varrho}_\nu$. But we cannot apply $D^{p.c}$ in order to decide whether so sufficiently large a circle has been obtained already or not yet, because point s is not available. So this important decision must be replaced by another one, as follows: with respect to any $\widehat{\varrho}_\mu$ occurring in the chain we continue:

$$S_1\,[P_5\{C_1(\dot{c}_o, \widehat{\varrho}_\mu), C_2(\dot{d}, \widehat{\varrho}_\mu)\}] = g;$$
$$S_2\,[P_4\{C_2(\dot{g},\widehat{\varrho}_\mu), L_1^{(o)}(d,e)\}] = f \textit{ so that } f \neq d;$$

so f is a point on the line through d parallel to $L_1(a,b)$ and so that $\widehat{fg} = \widehat{gd} = \widehat{gc} = \widehat{\varrho}_\mu$. *Now apply* $D^{l.c}$ *to* $L_1(a,b)$ *and* $C_1(\dot{c},f)$.

If they touch, then s is incident with the circle $C_1(\dot{c}_o, \widehat{\varrho}_{2\mu})$, and if they intersect, then s is inside $C_1(\dot{c}_o,\widehat{\varrho}_{2\mu})$. In both of these cases $\widehat{\varrho}_\mu$ *is sufficiently large*; there are two different continuations of our construction chain that will follow presently. Then $\widehat{\varrho}_\mu$ will be called $\widehat{\varrho}_\nu$. If the line and the circle do not intersect, then s is outside the circle $C_1(\dot{c}_o,\widehat{\varrho}_{2\mu})$; the radius $\widehat{\varrho}_\mu$ is too small and we must try a larger one.

(**a**) *$L_1(a,b)$ and $C_1(\dot{c},f)$ touch.*

$P_4\,\{C_1(\dot{c},f)\; L_1^{(o)}(a,b)\} = h$; if $L_1^{(o)}(a,b)$ and $L_1(c,d)$ are normal, h is the point of intersection required, but in that case we could also have found the point without the use of the Archimedean construction chain $(P_1{}'^{*})$. Next we suppose that the lines are not normal.

$P_5\,\{C_1(\dot{c},\, \widehat{\varrho_i}),\, C_2(\dot{h}, \widehat{\varrho_i}) = k_1\,\&\, k_2$; we will have to select the point k from k_1 & k_2 so that the triangles c,g,f and c,k,h have similar orientation. We may replace this decision with regard to the orientation of triangles by a *construction chain* (that does not depend on decisions of order!) *having a unique result*; the triangle c,k,h is constructible uniquely by a series of reflections just as used in the construction for *rotation* in congruence geometry. (Chapter V, 8, p. 80).

Finally: $Refl_k(c) = s$, completing the construction of the point s.

(**b**) *$L_1^{(o)}(a,b)$ and $C_1(\dot{c},f)$ intersect.*

$S_1\,[P_4\,\{C_1(\dot{c},f),\, L_1^{(o)}(a,b)\}] = h_i,\ i = 1{,}2$ (Double selection, cf. p. 68);
$P_5\,\{C_1(\dot{c}, \widehat{\varrho_v}),\, C_2(\dot{h}_i, \widehat{\varrho_v})\} = k_i^{(1)}\,\&\, k_i^{(2)}$;

now we shall have to select point k_i from $k_i^{(1)}\,\&\, k_i^{(2)}$ in such a manner that the triangles c,k_i,h_i and c,g,f have similar orientation. In an analogous way as mentioned in (*a*) this decision operation D^{ort} can be replaced by a construction chain (that does not depend on $\gtrless$-decisions) having a unique result; we will not repeat the process.

Finally: $Refl_{L_1^{(o)}(k_1,k_2)}(c) = s$, being the point of intersection required.

Consequently it has been shown that, in the Archimedean construction geometry (E',M), the operations of V are performable. So we find: $E'_M(B) = E'_V(B)$.

Remark. In the preceding we had to assume that the analytical image of E' was a geometry over an ordered field: the construction chain for the operation $P_1{}'$ (point of intersection of two non-parallel lines) depends essentially on a $\gtrless$-decision operation. This means that it may fail in the case that B contains *free points* that are non-decidable with respect to the $\gtrless$-relations between them and certain other objects of $E'_M(B)$. The application of ARCHIMEDES' axiom contains either a process of trial or a calculation process with regard to *necessary $\gtrless$-decisions*. If we can show – and we will do – that the possibility of $\gtrless$-decision operations is necessary for the validity of the MOHR-MASCHERONI theorem, this implies that the theorem only may hold good in *ordered construction fields. So the free adjunction operation A^f may not be admitted.* Given a basis B, only such a point t may be adjoined to B, forming $B' = [B,t]$, so that the $\gtrless$-decisions are applicable for all objects of the adjunctionless construction field $E'_M(B)$. With regard to the adjunction of the first two points the concept of free adjunction is senseless.

According to the result of theorem 26 we may conclude: The adjunctionless construction field $E'_M(B)$ does not admit of automorphisms leaving the basis pointwise fixed, but identity, except in the case that B consists of collinear points; then there are two automorphisms leaving B pointwise fixed, viz. *identity* and *reflection with respect to the line through the basis points.*

3. Necessity of ARCHIMEDES' axiom for the validity of the MOHR-MASCHERONI theorem in an adjunctionless construction field $E_M(B)$.

In this section it will be shown that in a non-Archimedean geometry E'' a basis consisting of three non-collinear points can be found, so that the adjunctionless construction field $E''_M(B)$ is a proper subset of the adjunctionless construction field $E''_V(B)$ under the operations of the ruler-and-compasses operation set V (Chapter VII); so we shall generally find: $E''_M(B) \neq E''_V(B)$.

Definition. *A point t of E'' is Archimedean over an ordered point couple a_0,a_1 if and only if it is inside at least one of the circles occurring in the Archimedean construction chain upon a_0,a_1.*

Lemma. *Let a,b,c,d be four distinct points in E'' so that the point of intersection s of the lines $L_1^{(o)}(a,b)$ and $L_1^{(o)}(c,d)$ is constructible from these points with the aid of the operations of M, then two points p_0,p_1 can be selected out of a,b,c,d so that s is Archimedean over the point couple p_0,p_1.*

Dem. The point set a,b,c,d will be called B_0; select p_0,p_1 from B_0 so that the distance $\widehat{p_0p_1} = \widehat{\varrho}$ is the *maximal distance* occurring in B_o. ($D^{p \cdot c}$) For the sake of convenience we shall call $p_1 = q_o$.

Describe all possible circles with both centre and circumference point in B_o and determine their points of intersection. Moreover, reflect the points p_o and q_o, the one with respect to the other:

$Refl_{q_0}(p_0) = p_1$; $Refl_{p_0}(q_0) = q_1$.

The point set B_1 will consist of B_0 and all new points thus obtained. $\widehat{p_1q_1} = 3\widehat{\varrho}$ is the maximal distance in B_1.

Thus we continue: B_{i+1} will consists of the points of B_i, the points of intersection of all circles with both centre and circumference point in B_i, and the points $Refl_{q_i}(p_i) = p_{i+1}, Refl_{p_i}(q_i) = q_{i+1}$. $\widehat{p_iq_i} = 3^i\widehat{\varrho}$ is the maximal distance in B_i. This means that each point of B_i is inside the circle with centre p_0 and radius $3^i\widehat{\varrho}$. Such a circle coincides with one of the circles occurring in the Archimedean construction chain upon p_0,q_0.

So if point s is constructible from B_0 in a finite number n of steps when using circles only, it must be inside the circle $C_2(p_0, 3^n\widehat{\varrho})$ in any case. We find that s is Archimedean over the point couple $p_0,p_1(=p_0,q_0)$, proving the lemma.

Theorem 29. *In a non-Archimedean ordered geometry E'' a basis B consisting of fixed points can be found so that the ordered adjunctionless construction field $E''_M(B)$ contains two distinct non-parallel lines $L_1^{(o)}(a,b), L_1^{(o)}(c,d)$, while the point of intersection s does not belong to $E''_M(B)$.*

Dem. Let B consist of three points, $z(0,0)$, $e_y(1,0)$, $t(\alpha,\beta)$ so that t is not Archimedean over the point couple z,e_y. (Coordinates with respect to an orthogonal cartesian coordinate system.) The y-axis (written $\bar{x}^{(o)}$) is constructible without the use of ARCHIMEDES' axiom (operation $L_3^{(o)}$ (inc z, $\perp \bar{y}^{(o)}$).

We have $n < \sqrt{\alpha^2+\beta^2}$ for all n; this means either $n < |\alpha|$ for all n, or $n < |\beta|$ for all n, or both. We suppose $n < |\beta|$ for all n, and $\alpha \neq 0$.

Next the points $p(0,\beta)$ and $q(1,\beta-\beta^{-1})$ will be constructed without the use of ARCHIMEDES' axiom, and we will show that the point of intersection s of $L_1^{(o)}(z,e_y) = \bar{y}^{(o)}$ and $L_1^{(o)}(p,q)$ does not belong to $E''_M(B)$, so that s is not constructible when using compasses only. (Fig. 12).

$P_1'^* \{L_2^{(o)}(\text{inc } t, // \bar{y}^{(o)}), \bar{x}^{(o)}\} = p(0,\beta)$;
$P_1'^* \{L_2^{(o)}(\text{inc } t, // \bar{y}^{(o)}), L_3(\text{inc } e_y, \perp \bar{y}^{(o)})\} = r(1,\beta)$;
$S_1\ [P_5\{C_1(e_y,r), C_1(r,p)\}] = d_1$ *and* d_2; these points are real because $\frac{1}{2} < |\beta|$; $Refl_{L_1(o)(d_1,d_2)}(r) = q(1,\beta-\beta^{-1})$.

The assumption that the point of intersection $s(\beta^2,0)$ of $\bar{y}^{(o)}$ and $L_1^{(o)}(p,q)$ would be constructible in the adjunctionless construction field $E''_M(B)$ will lead to a contradiction. For in that case s would be archimedean over the point couple p, e_y, these being the points with the greatest of the six distances between the four points z, e_y, p, q; consequently there would exist a natural number n_0 so that $n_0 \widehat{pe_y} > \widehat{zs}$, or $n_0 . \sqrt{1 + \beta^2} > \beta^2$. From this it would follow that $|\beta| < n_0 \sqrt{\beta^{-2} + 1} < 2\, n_0$, contradictory to the condition $n < |\beta|$ for all natural n. We find that s is not constructible from the points z, e_y, t when using circles only, proving the theorem.

Corollary. Theorem 29 shows that in Euclidean geometry ARCHIMEDES' axiom is necessary for the validity of the MOHR-MASCHERONI theorem in its rigid form: a basis containing two distinct points once given, no further use will be made of adjunction operations. Moreover we find that the adjunction of free points is not allowed when adjoining the basis points.

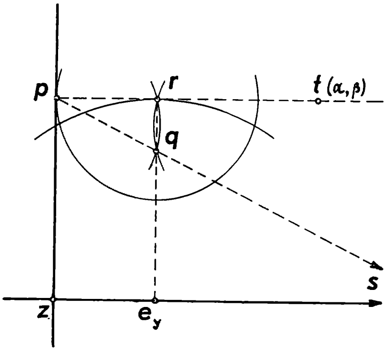

FIG. 12

4. Special construction fields in non-Archimedean geometry E''.

Let V be the Euclidean ruler-and-compasses operation set defined in Chapter VII; M'' will be the operation set M in which a certain *enumerable* series of determinate adjunction operations is admitted in order to construct objects of $E''_V(B)$, provided that a finite number of these adjunctions is sufficient to construct an object of $E''_V(B)$, for an arbitrary basis B in E'' So we consider non-Archimedean construction geometries with the property: $E''_V(B) = E''_{M''}(B)$, for every B. (E'' will be an *ordered* geometry). Although adjunction operations are essentially used, it is not correct to say that $E''_{M''}(B)$ is an adjunctive construction field; the coordinates and equation coefficients of the objects of $E''_{M''}(B)$ are no functions of the coordinates of the adjoined points, but the possibility of the construction of a certain object of $E''_{M''}(B)$ is not independent of the adjoined points.

Let B_0 be a basis in E''; let $s_\lambda(\lambda \in \Lambda,\ \Lambda$ an enumerable index system) be the enumerable point set in E'' the successive adjunction of which enables us to construct all points of $E''_V(B_0)$, each object of $E''_V(B_0)$ requiring at most a finite number of these adjunctions. Consider the series of bases B_0, $B_1 = [B_0, s_1], \ldots,$ $B_{i+1} = [B_i, s_{i+1}],$ and the corresponding series of (ordered) adjunctionless construction fields
$E''_M(B_0) \subset E''_M(B_1) \subset E''_M(B_2) \subset$ then it follows that $E''_V(B_0) \subset \bigcup_{\lambda\in\Lambda} E''_M(B_\lambda)$.

Definition. Let a be any (fixed) point of E'' *An enumerable series of points* $t_1, t_2, t_3,$ *with the property* $\overset{\frown}{at_i} < \overset{\frown}{at_{i+1}}$ *for every* $i > i_0(a)$ *will be called a majorant range in* E'' *if and only if for every point* p *at least one* t_ν *can be found so that* $\overset{\frown}{ap} < \overset{\frown}{at_\nu}$.

Theorem 30. *Let* B *be an arbitrary basis in* E''; *if* E'' *contains a majorant range* t_λ, $\lambda \in \Lambda$, *then* $E''_V(B) = E''_{M''}(B)$.

Dem. The only operation of V that may give any difficulty in the construction geometry (E'', M'') is P_1': point of intersection of two lines. In the derivation given in the demonstration of theorem 28 (p. 105) the cardinal point was the construction of so large a circle with fixed given centre, that its points of intersection with a fixed given line were real. Without repeating the entire proof we point out that such a sufficient large circle must occur when adjoining successively the points $t_1, t_2,$ of the majorant range of E'' to the basis B. Then we can use this circle for the circle $C_1(c_0, \overset{\frown}{\varrho_\nu})$ in order to complete the construction of the point of intersection s of the two lines, as described in the demonstration of theorem 28.

Consequently all operations of V are performable in a construction field $E''_{M''}(B)$.

Remark. Because the constructions in $E''_{M''}(B)$ depend essentially on decisions of order, the adjunction operations admitted are restricted to determinate adjunctions; free adjunctions are not admitted.

SAMENVATTING

In dit proefschrift wordt een symbolische schrijfwijze gebezigd voor verschillende constructieve operaties, die toegepast worden in de elementaire vlakke meetkunde. Hierbij wordt gedacht aan de mogelijkheid om voor een meetkunde een axiomastelsel op te bouwen bestaande uit drie soorten axiomas, nl. 1°) axiomas die een uitspraak doen t.a.v. de existentie van zekere objekten (punten, lijnen, cirkels,), 2°) axiomas die een uitspraak doen t.a.v. de existentie van zekere *relaties* (coincidentie, incidentie, evenwijdigheid, congruentie, loodrecht zijn, liggen tussen, liggen binnen, . .), 3°) axiomas die kunnen worden opgevat als zekere "rekenregels" voor de geïntroduceerde objecten en relaties. Met de in (1°) en (2°) genoemde axiomas zouden dan zekere constructieve operaties ingevoerd worden, zoals de *actieve operaties* (bijv.: een operatie die aan twee verschillende punten een lijn – de verbindingslijn – toevoegt), de *beslissingsoperaties* (bijv.: een beslissings-operatie die vaststelt of twee punten al dan niet verschillend zijn), en de *adjunctie-operaties* (bijv.: een operatie, die aan een zekere figuur "een of ander" punt toevoegt).

Nu zijn axiomatische beschouwingen in de studie niet opgenomen, ten eerste niet, om de stof enigszins te beperken, en ten tweede niet, omdat het werk, dat toch reeds twee verschillende categorieën van problemen behandelt, dreigde onoverzichtelijk te worden. Mogelijkerwijs zal een moderne meetkundige axiomatiek, als boven aangegeven, gebruik kunnen maken van de onderzoekingen hier gedaan naar eventuele afhankelijkheden van de constructieve operaties.

Kan men genoemde onderzoekingen als "meetkundig" betitelen, de tweede groep heeft meer een "analytisch" karakter; de laatsten hebben betrekking op de structuur van de *constructievelden*: een constructieveld is dan de verzameling van punten, lijnen, cirkels, bereikbaar, uitgaande van een *basis* van gegeven objekten, met behulp van de in een meetkunde ingevoerde constructieve operaties.

Het was gewenst, een serie van meetkunden te beschouwen, gerangschikt naar toenemend aantal ingevoerde begrippen, of constructieve mogelijkheden. Een merkwaardig onderscheid treedt aan de dag tussen constructievelden in de zg. spiegelingsmeetkunde (Hfdst. IV), afhankelijk van de algebraische structuur van de basis.

Bij de stap van spiegelingsmeetkunde naar congruentiemeetkunde worden de vertakkings-operaties ingevoerd, die de snijpunten van lijn en cirkel, of van twee cirkels voortbrengen; het ordebegrip speelt in de congruentiemeetkunde geen rol; derhalve mogen alleen die vertakkingsoperaties gebruikt worden, welke niet aan ongelijkheidsvoorwaarden gebonden zijn. In deze constructiemeetkunde wordt men geleid tot *automorfismen* van zekere constructievelden, die de basis puntsgewijze invariant laten. Deze automorfismen vormen een groep, die isomorf is, hetzij met de groep van Galois van een zeker kwadratisch uitbreidingslichaam van het coordinatenlichaam (op de coordinaten der basispunten), hetzij met het direct product van deze groep en een tweecyclische groep, die correspondeert met de permutatiegroep der isotrope punten.

De kwestie welk van de twee gevallen zich voordoet, hangt in het algemeen af van dezelfde algebraische structuur van de basis, die het genoemde onderscheid in de "orthogonaal-lineaire" constructievelden veroorzaakt. Zodra het ordebegrip zijn intrede doet, vallen de eerstgenoemde automorfismen (die dus corresponderen met genoemde groep van Galois) weg, behalve het identieke automorfisme; van de beide automorfismen corresponderend met de permutatiegroep der isotrope punten blijft slechts de identiteit over, indien het constructieveld "orienteerbaar" is. Deze orienteerbaarheid kan ook in constructievelden in niet-geordende meetkunden (constructief) worden gedefinieerd (Cf. Spiegelingsmeetkunde en congruentiemeetkunde).

In het laatste hoofdstuk wordt de constructiemeetkunde van MOHR-MASCHERONI in het Euclidische vlak behandeld. In het bijzonder wordt bewezen dat het axioma van ARCHIMEDES noodzakelijk is voor de geldigheid van de bekende stelling van MOHR-MASCHERONI: alle Euclidische (lineaire en kwadratische) constructies zijn uitvoerbaar met het gebruik van de passer alleen (afgezien van het trekken van lijnen). Om een analoge stelling in een niet-Archimedische (ofschoon geordende) meetkunde te verkrijgen, wordt de mogelijkheid van een aftelbare serie speciale adjunctie-operaties gepostuleerd.

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B. L. VAN DER WAERDEN.

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LIST OF SYMBOLS

Points	$a, b, \quad , t, p,$	Coincident	$=$
Lines	$\bar{a}, \bar{b}, \quad , \bar{u}, \bar{p},$	Distinct	$\neq$
Circles	$\bar{\bar{a}}, \bar{\bar{b}}, \quad , \bar{\bar{t}},$	Incident	inc
Distances	$\widehat{ab}, , \quad .\widehat{\varrho}$	Excident	exc
Angles	$a\hat{b}c, \hat{\varphi}, \quad , \bar{a}\,\hat{}\,\bar{b},$	Parallel	$//$
Congruent	$\simeq$	Normal	$\perp$

Oriented triangle	$\overrightarrow{a,b,c}$
Between (b between a and c)	$/a,b,c/$
Coordination of equivalent objects	&
Geometry	$G, P, A, R, C, E,$
Active operations	$P_1, P_1', \quad ., L_1, L_2, \quad , C_1,$
0-decision operations	$D^{pp}, D^{pc}, D^{par},$
$\gtrless$-decision operations	$D^{p \cdot p \cdot p}, D^{p \cdot c}, D^{l \cdot c},$
Selection operation	S_1
Adjunction operations	A^g, A^d, A^f
Operation sets (adjunctionless)	$V, L, Q, N,$
Operation sets (adjunctive)	$V(A), L(A), U(A),$
Basis of construction field	B, B_0, B'
Construction field (adjunctionless)	$G_V(B), R_L(B), E'_M(B'),$
Construction field (adjunctive)	$G_{V(A)}(B), A_{L(A)}(B'),$
Construction geometry	$(G,V), (R,L), (E,M),$

LIST OF OPERATIONS USED

Active operations

$P_1(\bar{a},\bar{b})$	point of intersection of lines $\bar{a}$ and $\bar{b}$	22
$P_1'(\bar{a},\bar{b})$	point of intersection of non-parallel lines $\bar{a}$ and $\bar{b}$	28
$P_1'^{*}(\bar{a},\bar{b})$	point of intersection of normal lines $\bar{a}$ and $\bar{b}$	102
$P_2(\bar{a},\bar{\bar{a}})$	"other" point of intersection of line $\bar{a}$ and circle $\bar{\bar{a}}$, both incident with a given point a	42
$P_2'(\bar{a},\bar{\bar{a}})$	idem, where $\bar{a}$ is incident with the centre of $\bar{\bar{a}}$.	100
$P_3(\bar{\bar{a}},\bar{\bar{b}})$	"other" point of intersection of circles $\bar{\bar{a}}$ and $\bar{\bar{b}}$, both incident with a given point a	43
$P_4(\bar{a},\bar{\bar{a}})$	both points of intersection of line $\bar{a}$ and circle $\bar{\bar{a}}$	91
$P_4'(\bar{a},\bar{\bar{a}})$	idem, where $\bar{a}$ is incident with the centre of $\bar{a}$	61
$P_4''(\bar{a},\bar{\bar{a}})$	idem, where $\bar{a}$ is excident with the centre of $\bar{\bar{a}}$.	103
$P_5(\bar{\bar{a}},\bar{\bar{b}})$	both points of intersection of circles $\bar{\bar{a}}$ and $\bar{\bar{b}}$	91
$P_5'(\bar{\bar{a}},\bar{\bar{b}})$	idem, where $\bar{\bar{a}}$ and $\bar{\bar{b}}$ are incident with their mutual centres	62
$M(a,b)$	midpoint of the point couple a,b	33
$Refl_{b}(a)$	reflection of point a with respect to b	41
$Refl_{\bar{a}}(a)$	reflection of point a with respect to line $\bar{a}$	41
$L_1(a,b)$	line passing through a and b	22
$L_2(\text{inc } a, // \bar{a})$	line through point a parallel to line $\bar{a}$	28
$L_3(\text{inc } a, \perp \bar{a})$	line through point a normal to line $\bar{a}$	42
$L_3'(\text{inc } a, \perp \bar{a})$	idem, where a is incident with $\bar{a}$	102
$L_3''(\text{inc } a, \perp \bar{a})$	idem, where a is excident with $\bar{a}$	102
$C_1(\dot{a},b)$	circle with centre a and circumference point b	41
$C_2(\dot{a},\widehat{bc})$	circle with centre a and radius $\widehat{bc}$.	43
$C_3(\dot{a},\widehat{\varrho})$	circle with centre a and fixed radius $\widehat{\dot{\varrho}}$	83

Decision operations

	0-decisions concerning:	
D^{pp}	coincidence or non-coincidence of two points .	22
D^{ll}	coincidence or non-coincidence of two lines	22

D^{cc}	coincidence or non-coincidence of two circles.	42
D^{pl}	incidence or excidence of point and line	23
D^{pc}	incidence or excidence of point and circle	42
D^{par}	parallelism or non-parallelism of two distinct lines	29
D^{norm}	normality or non-normality of two lines	42
D^{ort}	similar or opposite orientation of two isosceles right angled triangles	55

$\gtrless$-decisions concerning:

$D^{p\cdot p\cdot p}$	betweenness of three dictinct collinear points.	84
$D^{l\cdot c}$	intersection or non-intersection of line and circle	85, 92
$D^{c\cdot c}$	intersection or non-intersection of two circles	92
$D^{p\cdot c}$	the position of a point inside or outside a circle	85
$D^{p\cdot p\cdot l}$	the situation of two points on similar or opposite sides of a line	85
D^{ort}	similar or opposite orientation of two triangles	86

Selection operations

S_1	arbitrary selection out of a set of equivalent objects	10
S_2	combined operation of decision and selection	10

Adjunction operations

A^g	geometrical adjunction	12
A^d	adjunction of a determinate point	12
A^f	adjunction of a free point	13

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INSTYTUT MATEMAT. P.A.N.
BIBLIOTEKA

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STELLINGEN

behorende bij D. Kijne, Plane Construction Field Theory,
Utrecht 28 mei 1956.

I

Wanneer men passende constructieve operaties invoert in de n-dimensionale meetkunde R^n, kunnen de beschouwingen van dit proefschrift tot constructiemeetkunden in R^n worden uitgebreid.

II

De opmerking van BIEBERBACH, dat het door H. TIETZE gemaakte onderscheid tussen eenduidige en niet-eenduidige constructies in het Euclidische vlak te interpreteren is als een onderscheid tussen constructies in een georienteerd en in een niet-georienteerd vlak, is niet geheel juist.

L. BIEBERBACH, Theorie der geometrischen Konstruktionen, Basel 1952, pp. 151, 152.
H. TIETZE, Zie litteratuurlijst bij het proefschrift.

III

Het verdient de voorkeur om in de formulering van de stelling van BOTTEMA: „De Euclidische kwadratische constructies kunnen worden uitgevoerd met de lineaal indien een ellips (of een hyperbool) met twee brandpunten gegeven is", de aard van de kegelsnede in het midden te laten en de beslissingsoperatie, die vaststelt of twee lijnen al dan niet evenwijdig zijn, als postulaat te rekenen tot de met de lineaal uitvoerbare operaties.

O. BOTTEMA, On geometrical constructions with the ruler, Nieuw Archief voor Wiskunde (3) 3, 1—5 (1955).

IV

De door ALMERING ingevoerde meetkunde van toelaatbare punten en rechten is dezelfde als de meetkunde in de in dit proefschrift geintroduceerde orthogonaal-lineaire constructievelden op een basis bestaande uit drie punten met toelaatbare afstanden. De merkwaardigheid dat in deze meetkunde een tweewaardige constructieoperatie mogelijk is, terwijl in het orthogonaal-lineaire constructieveld alle operaties eenwaardig zijn, laat zich verklaren door het feit dat de bewuste tweewaardige operatie wordt toegepast op het object „hoek", dat niet in de gewone vorm bestaat in de spiegelingsmeetkunde, waar het orthogonaal-lineaire constructieveld juist zijn oorsprong vindt.

J. H. L. ALMERING, Rationaliteitseigenschappen in de Vlakke Meetkunde, Diss., Amsterdam 1950.

V

Het tweede gedeelte van het bewijs van AHLFORS voor de stelling: „Een familie F van analytische functies f in een domein Ω is normaal indien en slechts indien bij elke compacte verzameling $E \subset \Omega$ een constante M bestaat zodanig dat $|f'(z)| \leqslant M(1+|f(z)|^2)$ voor alle $z \in E$, en $f \in F$", steunt op een conclusie die niet door de geleverde argumentatie gerechtvaardigd wordt.

L. V AHLFORS, Complex Analysis, New York-Toronto-London 1953, pp. 169, 170.

VI

Voor verschillende moeilijkheden die BERGHUYS aan de orde stelt, als: het tweeledig karakter (inductief en deductief) van het wiskundig bouwen, met zijn consequenties voor de meetkundige axiomatiek, en het tweeledig karakter van het taalgebouw dat de meetkundige constructies begeleidt, kan wellicht een oplossing naderbij gebracht worden door op het verschil tussen meetkunde en constructiemeetkunde acht te geven

J. J. W BERGHUIS S. J. ,Grondslagen van de aanschouwelijke meetkunde, Groningen-Djakarta 1952 pp. 183—187.

VII

De tweede hoofdstelling van R. NEVANLINNA en de overeenkomstige stelling van L. V AHLFORS kunnen op elegante wijze geformuleerd worden met behulp van limieten volgens een filter.

R. NEVANLINNA, Eindeutig Analytische Funktionen Berlin 1936, pp. 240—241 en pp. 338—341.

VIII

In de intuitionistische wiskunde kan op grond van de hoofdstelling van L. E. J. BROUWER over finiete spreidingen bewezen worden: Heeft een groep G de voortbrengenden a en b, terwijl ieder element van G een eindige orde bezit, dan kan men een natuurlijk getal N bepalen zodanig dat ieder element van G een orde bezit kleiner dan N.

IX

Het is gewenst om enkele eenvoudige ontoegankelijkheidsconstructies op de middelbare school te behandelen.

X

Wanneer men bij het meetkundeonderwijs de exact-axiomatische methode enigszins vervangt door een intuitief-practische, is het gevaar niet denkbeeldig dat het verkrijgen van inzicht in het karakter van de wiskunde voor de leerlingen extra wordt bemoeilijkt.

XI

In de elementaire analytische (geordende) meetkunde kan men voor een algemeen gegeven echte driehoek $a\ (x_1,y_1)$, $b(x_2,y_2))$, $c(x_3,y_3)$ de respectieve binnen- en buitenbissectrices rechtstreeks bepalen.

XII

Het verdient dringend aanbeveling om formules als $\sqrt{x^2} = |x|$, $\sqrt{1 - \sin^2 x} = |\cos x|$, $\log ab = \log |a| + \log |b|$, $\log x^2 = 2 \log |x|$, enz., op de scholen te behandelen en geregeld te gebruiken.

XIII

De schrijfwijzen $\lim\limits_{x \to a}$ en $\lim\limits_{a \leftarrow x}$ voor resp. linker en rechter limiet zijn niet te verkiezen boven $\lim\limits_{x \uparrow a}$ en $\lim\limits_{x \downarrow a}$.

Cf. E. J. WASSCHER, Nieuw Leerboek der Algebra III, Groningen-Djakarta 1956.

XIV

Het lijkt waarschijnlijk dat de astronomen die zich bezig houden met het onderzoek van de Kleine Planeten binnen niet al te lange tijd zullen overgaan tot een nieuw werkprogramma, waarbij men zal afzien van het systematisch waarnemen van de zg. „niet-nuttige" Kleine Planeten.

Transactions of the International Astronomical Union, Vol. VIII, 1952.

XV

Het lijkt mogelijk om enige optische wetten die afgeleid kunnen worden uit de tweede hoofdwet der thermodynamica als bijzondere gevallen te zien van een meer algemene wet.

XVI

De didaktische moeilijkheden gelegen in de wijze waarop niet-mathematici soms afleiden dat zekere grootheden door een integraal kunnen worden voorgesteld (bij voorbeeld door het beschouwen van „oneindig kleine", of „zeer kleine" grootheden), behoeven niet door de docent in de wiskunde uit de weg geruimd te worden.

XVII

De nadelen verbonden aan het vrijstellingensysteem bij de eindexamens der H.B.S.-B zijn groter dan de voordelen.

XVIII

De belangrijke opvoedende waarde van het vak „tekenen" maakt het wenselijk dat dit vak als school- en als examenvak hoger wordt gewaardeerd dan thans gebruikelijk is.